PLANE GEOMETRY

SEYMOUR

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PLANE GEOMETRY

BY

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SEYMOUR'S PLANE GEOMETRY

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PREFACE

The addition of another text in Plane Geometry to the list of texts now in use in secondary schools cannot be made without explanation. To justify its appearance such a book should possess features not common to other texts but found by experience to be a decided help in the teaching of geometry. A mere innovation is no justification. Any departure from the traditional text should have a sound pedagogical and psychological basis. It should recognize the purpose in teaching geometry and be made to serve that purpose only.

Three features found in this book will be mentioned in detail. They are the result of fifteen years' experience in teaching geometry in secondary schools, in giving courses for teachers in normal school and college, and in observation of hundreds of classes in geometry. They have therefore been worked out not in theory but in experience, from which their practicability may be accepted as proved.

1. *An Introductory Course.* The purpose of this course is to acquaint the pupil in an interesting way with some of the fundamental tools of geometry, to lead him to discover in an informal manner some of the simpler geometric facts, and gradually to convince him of the necessity of a formal proof to make his conclusions trustworthy. Too often the pupil is plunged at once into Demonstrative Geometry before he is convinced of the necessity of such a course; or he is burdened with definitions, axioms, etc., of which no immediate use is seen or made. Lack of understanding and interest, so common at the outset, is often directly traceable to one or the other of these causes. The question "What

is the use of *proving* that — you can *see* it is so," expressed or implied, is seldom satisfactorily answered.

Just as the pupil about to start a course in Manual Training is not immediately set at work on a project but is taught the use of the individual tools that he will need, such as the plane, the saw, etc., so the pupil about to start a course in Plane Geometry should not be immediately set at work to prove certain statements but should be taught the use of the tools that he will need, such as rulers and compasses, as well as definitions and axioms. Also just as the pupil in manual training later learns how to select and coördinate his tools in making projects — choosing the proper tool and material at the proper time — so the pupil in geometry later will learn how to select and coördinate his tools so as to accomplish his project — proving some geometric statement to be true.

2. *The Analytic-Synthetic Method Used in Proving Propositions.* One of the main objects in teaching Demonstrative Geometry is to train pupils to discover proofs by *thinking naturally, logically,* and *systematically.* Manifestly the best way to do this is by questioning along the line one would naturally think in discovering the proof. The usual text in Plane Geometry records this proof synthetically, but seldom gives the pupil an inkling of how it was discovered. As he reads the proof, he is convinced of the *right* to take the various steps but not of the *reason* for taking them. He is continually asking, "How did they know that was the step to take?" or, "If I were given that proposition, how should I know enough to take that step?"

Throughout this text, except in those instances where the proposition is so simple that the proof is left entirely to the pupil, the author attempts, by an informal analysis, consisting of a series of natural and logical questions, to get the pupil to see how one would go about to discover the proof from the material immediately given and already at hand.

In the early propositions of Book One and in the case of those which are more than ordinarily important or difficult of proof, the analysis consists merely of a statement of the method to be used in the proof. In such cases a formal synthetic proof follows.

3. *Discussions after Propositions.* Special features or generalizations, where possible, are given in these discussions. It frequently happens that pupils study the proof of a proposition and leave it with a very narrow and often incorrect notion of the situation. For example, in proving that two triangles are congruent if their sides are respectively equal, the drawing usually shows the longest pair of equal sides in coincidence (some texts even tell the pupil to use these sides) and seldom is the pupil told in a discussion immediately following that any pair of equal sides may be matched. He therefore is led to believe that the longest pair of equal sides must always be used and hence he loses an interesting generalization and the opportunity to do some original investigation. Furthermore, in connection with the same proposition, since superposition is naturally the first method that comes to his mind, a discussion should also include the possibility and desirability of such a procedure. More good is likely to result from a thorough study of one proposition in the form of discussion, generalization, etc., than from a superficial *learning* of several propositions.

Attention might also be called to other features of the text that deserve mention. For example, definite instruction is given on how to attack exercises (§§ 129, 220) and problems in construction (§ 316). Frequently pupils approach such work with no systematic method of attack. Statements are made and lines are drawn that are permissible but quite irrelevant, in the hope apparently that if this procedure is continued, the correct method will eventually be reached. While there is no panacea for solving exercises, there are some general

suggestions that will give direction to the pupil's efforts, and these suggestions are attempted in this text.

In general, after each proposition a few exercises are given involving its immediate application. This will have a tendency to clinch the proposition. Important exercises are marked with an asterisk (*). These include not only those exercises which might be considered fundamental and important but also exercises which are used in proving other exercises in the text. Hence these exercises should be included in the minimum list worked out by any class.

Careful consideration has been given to the subject of loci in the hope that some of the difficulties which it usually presents may be anticipated and eliminated. The treatment has been deferred to the end of Book Two.

The parallel-column arrangement of proof is used because it seems to emphasize more strongly the necessity of giving a reason for each statement made. This systematic and orderly arrangement of the work will lead to clear and orderly thinking on the part of the pupil.

In keeping with the general feeling that the formal theory of incommensurables and limits is of little profit to the average pupil of geometry, and in harmony with the recommendations of the National Committee on Mathematical Requirements, the meaning of propositions involving this theory has been made clear by illustrations, and the pupil has been convinced of their truths with no attempt at proof.

The chief aim of this text is to make real discoverers out of pupils in geometry. This it tries to accomplish by teaching them how to think naturally, logically, and systematically and by giving them abundant opportunity to apply this teaching.

Grateful acknowledgment is made to Mr. Ralph P. Bliss of the Alexander Hamilton High School, Brooklyn, New York, for many helpful suggestions.

SUGGESTIONS TO THE TEACHER

In order to teach geometry successfully, at the outset two things are essential: (1) the teacher must be convinced that the subject is worth while, and, as a result, (2) he must teach with a definite purpose or end in view. The nature of the recitation and the quality of the results will in large measure be determined by this purpose.

In geometry it is not so much the facts that are learned as the way in which they are learned that is important. The purpose here should be to make of the mind not a storehouse for keeping facts but rather a factory for turning them out, with the world, not books, as the place to get the raw material. If this is true, we may set up as one of the main objects in a course in Demonstrative Geometry teaching pupils how to think *naturally*, *logically*, and *systematically* whenever they are given a task to perform. To this end several suggestions are here offered that may prove helpful in your efforts to attain this result.

1. Try to create in the minds of the pupils the spirit of discovery. Through careful questioning in every proposition or exercise, try to get the pupil to think along the lines on which one would naturally work in discovering the fact.

2. In preparing the development of the lesson, study carefully the analysis found in each proposition. Get the pupil to see the purpose and importance of working out the analysis before considering the formal proof.

3. Discourage getting proofs by memorizing the one in the text. All proofs should be obtained by an outline similar to that given in the analysis. The formal proof, where given, should be studied afterwards.

4. When a pupil is called on to give a formal proof, it is well, before the proof is begun, to have the pupil outline

briefly what his plan of procedure is to be. A pupil about to prove Proposition III, Book One, might say: "I am going to prove that angle A equals angle B by means of corresponding parts of congruent triangles, and I will form these triangles by bisecting angle C," etc.

5. Emphasize logical sequence as illustrated in the construction of a list of propositions for a text in geometry. Show that in some cases the sequence is merely one of order and might be changed, while in other cases the order is unchangeable.

6. Encourage originality. Do not let the pupil form the idea that the proof given in the book is the only one possible; but, when he submits an original proof, impress on him the fact that the logical sequence of propositions in his text must be observed if his proof is to be valid.

7. Review frequently the different ways of proving angles equal, lines equal, etc., so that facts learned will be arranged systematically in the mind of the pupil.

8. *Whenever proofs and necessary constructions are incomplete in the text, or when they are omitted or made without explanation, rigidly insist that they be given in complete form with reasons.* Statements should be given rather than referred to by article. Reasonable abbreviations should be permitted, however, in writing these statements.

9. Follow each proof with a discussion if it permits of any variation in procedure or in generalization.

10. While the pupil should be familiar with the terms *hypothesis* and *conclusion*, he should generally use the simpler terms *given* and *to prove*.

11. Make haste slowly at the outset. Be sure that the pupils understand the tools they have to work with and the real problem in geometry; and, before they start the formal work of Demonstrative Geometry, be sure that they are convinced of the necessity of a formal proof.

SUGGESTIONS TO THE PUPIL

You are about to start the study of Plane Geometry. It will be helpful to you to know in the beginning what this course purposes to do and what your part is to be in accomplishing this purpose. Both of these points may be made clear by a comparison between the work you are about to start and that of a carpenter.

A carpenter about to construct a building must first have a foundation on which to build as well as tools and material for the construction. If the work is to be a success, the carpenter must thoroughly understand how to use his tools and how to select the material in making the various parts of the building.

In your work in geometry you, too, are going to do some building for which you must have a foundation on which to work. This will be in the form of a list of foundation facts — facts which are simple and whose truth will be accepted without question. You must also have tools, such as the ruler and compasses, and you will be supplied with material arranged in groups and variously labeled, such as definitions and axioms. The work you are to do will consist in building up proofs for certain geometric statements. Whenever such a statement is given, it will be your task to discover what material should be selected from these various groups and how it should be put together to accomplish the task.

The building at first will be simple ; that is, you will readily see what material must be used to establish the given statements and how it should be put together. As you continue your work and the material at your disposal increases, the

work of selecting the proper material will doubtless call for greater exercise of judgment; but its nature will remain essentially the same — a task to be accomplished and a judicious selection of the right material to accomplish it.

Two things will be helpful to you when you have one of these tasks to perform:

(1) *A thorough familiarity with the use of the tools and of the material for selection.* The work given in the Introductory Course will give you this familiarity. You must make sure that you know how to use the ruler, compasses, etc., and that you understand all the foundation facts and definitions that are given.

(2) *A careful study, in connection with the work in Demonstrative Geometry, of the list of questions suggested with each geometric statement given you to prove.* It is the purpose of these questions to train you in a systematic way to select and arrange the material for the proof. If this is done thoroughly, you will gradually acquire a method of attack that will prove exceedingly helpful to you in discovering proofs.

Remember that the good you get out of your work in geometry, and consequently your success, will be measured directly by the ability and power you develop to discover proofs yourself. The facts that you learn have only incidental value; it is the way in which you learn these facts that counts.

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THE ORIGIN OF GEOMETRY

GEOMETRY is a very old science. The first record we have of its study is found in a manuscript written by Ahmes, an early Egyptian scholar, about 1550 B.C. This manuscript is believed to be a copy of a treatise which dated back probably more than a thousand years, and describes the use of geometry at that time in a very crude form of surveying or measurement. In fact, geometry, which means "earth measurement," received its name in this manner. From this early record we find that the Egyptians cared but little for theoretical results, their emphasis in geometry being placed on making constructions and determining areas.

Inasmuch as these early people concerned themselves solely with the physical applications of geometry, making up their rules from a number of observations, but giving no attention to proving that these rules were accurate, it was quite natural that some of the statements and formulas which they used should be only approximations to the truth. For example, one of their rules states that "the area of an isosceles triangle is equal to one half the product of the base and one side." The formula for the area of an isosceles trapezoid was also incorrectly given.

This attitude toward the study of geometry partially explains two of the great defects in Egyptian geometry:

(1) Its failure to construct a rigorously logical system of geometry based on a few fundamental facts. Many of the rules had probably never been proved, but were known from mere observation.

(2) Its inability to generalize into one statement a series of special cases. Some of the simplest geometric truths were divided into numberless special cases, each of which received special treatment.

When the Greek scholars from Athens, who disdained the study of a subject for any practical use that might be made of it, came to Egypt, the study of geometry assumed an entirely different aspect. These scholars concerned themselves solely in discovering why the statements and formulas used in measuring were true and what relations existed among the parts of various geometric figures which they drew for investigation and study.

The discoveries made by these scholars were later collected and compiled by Euclid (300 B.C.) in a book which he called his "Elements." The remarkable nature of this book is seen from the fact that it is the oldest scientific text still in use. Since 1482 over one thousand editions of this book have been made, the modern text in England being founded on an edition issued in 1758.

With the exception of the exercises and applications which modern texts in geometry contain, they differ but little in form and treatment from this book. With modern scholars the cultural aim still holds first place and the applications that can be made of the facts of geometry are considered as a secondary reason for its study.

SYMBOLS AND ABBREVIATIONS

The following symbols and abbreviations are used frequently in the work in Demonstrative Geometry.

$=$	equal, equals, equal to, is (*or* are) equal to	*adj.*	adjacent
$>$	is (*or* are) greater than	*alt.*	alternate, altitude
$<$	is (*or* are) less than	*Ax.*	axiom
$\cong$	congruent, is congruent with	*comp.*	complement, complementary
$\therefore$	therefore	*Cons.*	construction
$\perp$	perpendicular, perpendicular to, is perpendicular to	*Cor.*	corollary
$\perp$s	perpendiculars	*corr.*	corresponding
$\parallel$	parallel, parallel to, is parallel to	*Def.*	definition
$\parallel$s	parallels	*ex.*	exercise, exercises
$\sim$	similar, is (*or* are) similar to	*ext.*	exterior
$\frown$	arc; $\overset{\frown}{AB}$, *read* arc AB	*Fig.*	figure
$\angle$	angle; $\measuredangle$s angles	*hyp.*	hypothesis
$\triangle$	triangle; $\triangle$s triangles	*int.*	interior
$\square$	parallelogram; $\square$s parallelograms	*isos.*	isosceles
$\odot$	circle; $\odot$s circles	*med.*	median
. . .	and so on	*opp.*	opposite
a',	b' . . . a-prime, b-prime . . .	*Post.*	postulate
a_1,	a_2 . . . a-sub-one, a-sub-two . . .	*Prin.*	principle
		Prop.	proposition
		quad.	quadrilateral
		rt.	right
		st.	straight
		Sug.	suggestion
		sup.	supplement, supplementary
		Th.	theorem

The signs $+$, $-$, $\times$, $\div$ have the same meaning as in arithmetic. *a.s.a.*, having a side and the two adjacent angles of one triangle equal respectively to a side and the two adjacent angles of the other; *s.a.s.*, having two sides and the included angle of the one triangle equal respectively to two sides and the included angle of the other; *s.s.s.*, having three sides of one triangle equal respectively to three sides of the other; *s.a.a.*, having a side and two angles of one triangle equal respectively to a side and two angles of the other.

PLANE GEOMETRY

PART ONE

AN INTRODUCTORY COURSE

GEOMETRIC FIGURES

1. Introduction. *Geometry* is the study that deals with the properties, measurements, and construction of flat figures, such as angles, triangles, and rectangles, and of solid figures, such as cubes, pyramids, and spheres. You are already familiar with many of the terms used in geometry, such as *angle*, *triangle*, and *circle*, and you have a general notion of their meaning. It will be well, however, for you to consider further some of the most important terms, in order that you may have a more accurate understanding of their use in geometry.

2. Solids. A block of wood, an iron bar, and a baseball are examples of physical solids. In physics we are interested chiefly in the properties, etc. of the material which constitutes physical solids, while in geometry we study the size, the shape, etc. of the solids but not the material of which they are made. If a block of wood is moved from any position to another, the space which it formerly occupied is a *geometric solid*.

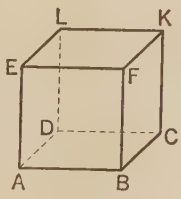


FIG. 1

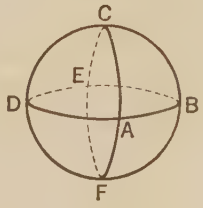


FIG. 2

3. Surfaces. In Fig. 1 each of the six faces which bound the solid is called a *surface*. Similarly, if Fig. 2 represents a sphere, *ABCDEF*, which limits the solid, is a surface. Likewise the boundaries of geometric solids are surfaces.

4. Lines. The intersection of two surfaces of a solid is called an edge. In a geometric solid, these edges are called lines. Hence a *line* may be considered as the intersection of two surfaces. Thus, the intersection of the bottom and front faces of Fig. 1, page 1, gives the edge or line AB.

5. Points. In Fig. 1, page 1, edges AB and CB meet at B, a vertex of the solid. In a geometric solid, the vertices are called points. Hence, in general, a *point* may be considered as the intersection of any two lines.

6. Magnitudes. Solids, surfaces, and lines are frequently referred to as *magnitudes* when their extent or size is considered; they are called *geometric figures* or merely *figures* when their form or shape is considered.

THE STRAIGHT LINE

7. The *straight line* is one of the most fundamental concepts in geometry. It is therefore difficult to define it. The following tests, however, will make clear its nature:

1. Carefully fold a piece of paper. The crease is a straight line.

2. Tie a weight to a string and suspend it from your hand. The string will have the appearance of a straight line.

3. Place your ruler flat on the paper. With the pencil point on the paper and firmly against the ruler move the pencil. If the edge of the ruler is "true," the mark will be a straight line.

4. Hold your ruler on a level with the eye, with one end of the ruler pointing toward the eye. If the edge is "straight," it will appear as a point.

For reasons evident from the above tests the instrument used to draw straight lines is called a *straight edge*. The ruler ordinarily used to draw straight lines is merely a straight edge divided into various units, such as the inch, the half

inch, etc. The ruler therefore may be used not only to draw straight lines but to measure them as well.

8. A *plane surface*, or, more briefly, a *plane*, is another one of the fundamental concepts of geometry difficult to define. The usual test states that a surface is a *plane* if the straight line joining any two of its points lies wholly within the surface. The following applications will make clear its meaning:

Ex. 1. The covers of a book, the walls of a room, the glass in windows are generally illustrations of planes. Can you find an illustration that is an exception to this statement and tell why it is an exception?

Ex. 2. Apply the edge of your ruler to the top of the desk in every direction. If the top of the desk is a plane, every part of the edge will touch the desk. Why is this so?

Ex. 3. How do workmen in finishing off concrete walks or in plastering a room determine with a straight edge if the surface is a plane?

Ex. 4. If the walls of your recitation room are taken as illustrations of planes, what is the intersection of two planes? This is sometimes given as a test for a plane.

We call the course which we are about to study *Plane Geometry* because all the figures we are to deal with are plane figures.

Exercises — Group 1

THE STRAIGHT LINE

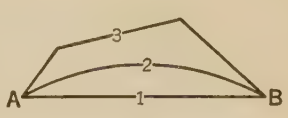
1. Note in the classroom several illustrations of the straight line. Mention other illustrations outside the school.

2. Tear a piece of paper. Then make the rough edge straight.

3. Through a point P, represented by a dot, draw five straight lines. How many straight lines can be drawn through P?

4. Select another point and call it A. Draw five straight lines each ending in A.

5. In moving from A to B in the drawing, which path is the shortest?



9. PRINCIPLE 1. *A straight line is the shortest line between two points.*

A straight line has the same direction throughout its length.

Unless otherwise stated, the word *line* will mean *straight line.*

Ex. 1. Why do we "cut corners" when in a hurry, instead of going around the corners?

Ex. 2. Represent two points C and D on your paper. Can you draw a straight line through C and D? Can you draw two straight lines through C and D? three straight lines? Can more than one line like 2 or 3 in ex. 5 above be drawn through C and D?

10. PRINCIPLE 2. *Only one straight line can be drawn between two points.*

This principle is often stated: *Two points determine (fix) a straight line.* If we attempted to draw two straight lines between the two points, they would turn out to be the same line; that is, they would coincide.

11. A line like 3 in ex. 5 above, which is made up of different successive straight lines, is called a *broken line.* A line like 2, no portion of which is straight, is called a *curved line*, or a *curve.*

Ex. 1. How can Prin. 2 be used to test the edge of your ruler? to rule a margin on your paper? to make a picture hang "level"?

Ex. 2. Why is it that when you sight an object and walk toward it, you usually walk in a straight line?

Ex. 3. Why does a rifle barrel have two sights?

Ex. 4. Draw two straight lines that cross each other. In how many points do they cross? Draw these two lines in various positions. In how many places can two straight lines cross; that is, how many points may two *different* straight lines have in common?

12. PRINCIPLE 3. *Two straight lines can intersect in but one point.*

If the two straight lines crossed at both A and B, what would be true, according to Prin. 2?

Would this in any way contradict the statement in Prin. 3?

If, then, we accept Prin. 2, what conclusion must we make about Prin. 3?

Ex. 1. If the streets of a city are straight lines and the crossings are points, show how Prin. 2 and Prin. 3 might be illustrated.

Ex. 2. If you know that a boy sits in a certain aisle and in a certain row, does it follow that you know in exactly what seat he sits?

Ex. 3. Show how Prin. 3 is used when we read a railroad timetable.

13. Line Segments. The lines used so far have represented lines of unlimited length; but in drawing them we used only a limited portion of the lines. A limited portion of a line is called a *line segment* or simply a *segment.* Where there is no ambiguity, it is customary to use the word *line* instead of *segment.*

A line may be represented, as in the drawings, either by lettering its end points with capital letters and reading it "line AB" or by writing a small letter on the line and reading it "line a."

A line segment may be extended indefinitely through both end points.

14. Equal Segments. *Two line segments are equal* when their end points can be made to coincide.

15. *A line segment is bisected* by a point when it is divided into two equal parts by the point. The point is called the *mid-point.*

Exercises — Group 2

LINE SEGMENTS

1. Draw lines of the following lengths: $3''$, $2\frac{1}{2}''$, $1\frac{7}{8}''$, $3\frac{3}{4}''$.

2. Draw several lines in different positions and of different lengths. Estimate the length of each line. Measure the line to the nearest unit on your ruler and note the error in your estimate.

SUG. Lines are best measured by means of compasses. Adjust the compasses so that the points rest on the ends of the line. Then, without changing the opening of the compasses, place the end points on the ruler and note the distance between the points. Lines may be transferred in the same way.

3. Repeat ex. 2 until the lengths of lines can be estimated with reasonable accuracy.

4. Draw without aid of the ruler (free-hand) lines as nearly as possible of the following lengths: $4''$, $1''$, $2\frac{1}{2}''$, $3\frac{1}{4}''$. Measure the lines and note the errors that you have made.

5. Repeat exercises like those under ex. 4 until lines of various lengths can be drawn free-hand with reasonable accuracy.

6. Draw a line. Estimate its mid-point. Measure the line and note your error. Repeat this exercise until the mid-points of lines can be located with reasonable accuracy.

7. Repeat ex. 6 for the *trisection* point of a line.

8. Draw a line and try to make five others of the same length in various positions. Note the error each time.

Exercises — Group 3

USE OF THE EQUATION

1. Divide a line $4''$ long into two parts so that one part shall be $1\frac{1}{2}''$ longer than the other.

SUG. First find the length of each part by the use of the equation. Then accurately locate the division point in the line and mark the two segments.

2. Divide a line $5''$ long into two parts so that one part shall be (*a*) $2\frac{1}{4}''$ shorter than the other; (*b*) 3 times the other.

3. Divide a line $5\frac{1}{4}''$ long into parts whose ratios are $1:2:3$.

4. Divide a line $7''$ long into three parts so that each part is $1''$ longer than the one before it.

5. Divide a line $6\frac{3}{4}''$ long into three parts so that each part is $\frac{1}{4}''$ shorter than the one before it.

6. If we represent the length of a line by a, how may we represent the length of a line that is $3''$ longer? If the combined length of the two lines is $15''$, what is the length of each line?

Exercises — Group 4

Drawing to a Scale

1. If a line $1''$ long is to represent one $8'$ long, draw lines to represent the following distances: $16'$, $12'$, $20'$, $11'$, $19'$, $8\frac{1}{2}'$, $5'\ 3''$.

2. If $\frac{1}{4}''$ represents $1'$, what distances are represented by lines of the following lengths: $2''$, $5''$, $1\frac{1}{2}''$, $2\frac{3}{4}''$, $4\frac{7}{8}''$?

3. The distance between New York and San Francisco is about 3200 miles. How far apart would these two places be on a map drawn to a scale of $1''$ to 200 mi.?

4. If $\frac{2}{3}''$ on a map corresponds to an actual distance of 9 miles, what distance on the map will represent 21 miles?

5. The following table gives the approximate population of a certain city for various decades. Selecting a suitable scale, represent by a series of lines the various population readings.

1870 . .	15,000	1890 . .	33,000	1910 . .	48,000
1880 . .	23,000	1900 . .	38,000	1920 . .	60,000

6. If the scale is $\frac{1}{8}''$ to the inch, draw lines to represent the length and the width of the paper on which you are working.

7. Selecting a suitable scale, draw lines on your paper to represent (*a*) the width of the classroom; (*b*) the length of the classroom.

16. Addition and Subtraction of Lines.

Ex. 1. On a given line mark off a distance equal to a given line. Repeat for various lines.

Sug. This should be done by means of compasses.

Ex. 2. Draw two lines. On a third line mark off a line equal to their sum.

Ex. 3. Draw a line equal to the sum of three given lines.

Ex. 4. Draw a line equal to the difference between two given lines.

17. Equalities and Inequalities. Two magnitudes a and b may stand in any one of three relations to each other:

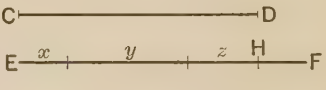
a may be equal to b.
a may be greater than b.
a may be less than b.

These three relations are represented in order as follows: $a = b$, $a > b$, $a < b$. The first is called an *equality* or an *equation;* the second and third are called *inequalities.*

Ex. 1. Represent by a line each of the following expressions in which a, b, and c represent line segments such that $a > b$ and $b > c$:

$$a + b + c;\ a + (b - c);\ a - (b - c);\ 2a;\ 3a - 2b.$$

Ex. 2. Take any line AB and divide it by any two points into three parts x, y, z. (See drawing.) Construct a second line CD equal to AB. (Use compasses.) On a third line EF lay off a line EH equal to $x + y + z$. Compare EH and CD. Compare EH and CD with AB.



18. Principle 4. *Quantities (lines) equal to the same quantity (line) are equal to each other.*

Show how ex. 2, § 17, illustrates this principle.

19. Principle 5. *The whole is equal to the sum of all its parts.*

What statement made in answering ex. 2, § 17, illustrates this principle?

Ex. In ex. 2, § 17, compare the line represented by $x + y$ with AB.

20. Principle 6. *The whole is greater than any of its parts.*

Ex. Select four lines a, a' (read "a prime"), b, b', such that $a = a'$, $b = b'$, and $a > b$. (See drawing.)

If $a = a'$
and $b = b'$
show that $a + b = a' + b'$

Transfer and compare the two sums by means of compasses.

21. PRINCIPLE 7. *If equals are added to equals, the sums are equal.*

Illustrate this principle by using numbers.

Ex. Show that $a - b = a' - b'$.

22. PRINCIPLE 8. *If equals are subtracted from equals, the remainders are equal.*

Illustrate this principle by using numbers.

Ex. If $a = a'$
show that $2a = 2a'$

23. PRINCIPLE 9. *Doubles of equals are equal.*

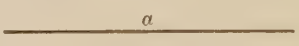
Illustrate this principle by using numbers.

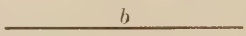
Ex. If $a = a'$
show that $\frac{a}{2} = \frac{a'}{2}$

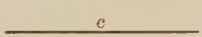
24. PRINCIPLE 10. State in words the principle just illustrated.

Illustrate this principle by using numbers.

Ex. Draw three lines a, b, and c, so selected that $a > b$ and $b > c$. Without measuring the lines, tell what relation a should have to c. Now with compasses compare a and c.







25. PRINCIPLE 11. *If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.*

Illustrate this principle by using numbers.

Exercises — Group 5

In each of the following exercises, state the principle used:

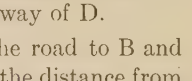
1. If it is 143 mi. by rail from New York to Albany, 147 mi. from Albany to Syracuse, and 80 mi. from Syracuse to Rochester, how far is it from New York to Rochester over the same rail?

2. The distance from A to B is 143 mi., and from B to C on another road, 57 mi. The distance from A to D is 143 mi., and from D to E on another road is 57 mi. Compare the distance from A to C by way of B with the distance from A to E by way of D.

3. In ex. 2, if it is 70 mi. from A to M on the road to B and 70 mi. from A to N on the road to D, compare the distance from M to B with the distance from N to D.

4. Is it farther from Boston to Albany or from Boston to Buffalo? Explain.

ANGLES



26. An *angle* is the figure formed by two straight lines drawn from the same point.

Lines AB and AC are called the *sides* or *arms* of the angle. The point A is called the *vertex* (plural, *vertices*) of the angle.

27. Notation. There are three common ways of naming angles, shown in the accompanying drawings:

FIG. 1

FIG. 2

FIG. 3

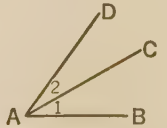


FIG. 4

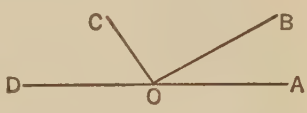
If there is but one angle, it may be designated by a capital letter written at the vertex, as angle A (written $\angle A$), Fig. 1, or by a small letter or a numeral written in the opening of the angle, as $\angle a$, Fig. 2 or $\angle 1$, Fig. 3. When there are several angles at the vertex, the angle in question may be designated by three capital letters, one on each arm and one at the vertex, *the vertex letter always being written between the other two letters*, as $\angle BAC$, Fig. 4.

28. Equal Angles. Two angles are said to be *equal* if they can be made to coincide.

The line through the vertex of an angle which divides it into two equal angles is called the *bisector* of the angle.

Exercises — Group 6

1. Name all the angles (six) in the drawing at the right.



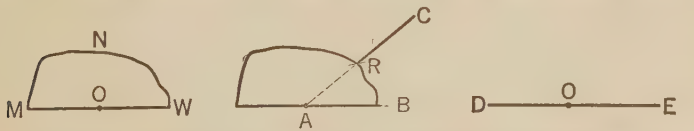
2. Through a given point draw four lines. How many angles are formed? Letter and read all the angles.

3. Draw an angle BAC. Extend the sides AB and AC of the angle. Does this change the size of the angle?

4. Draw on paper an angle BAC. Place your pencil flat on the paper directly over arm AB and with the rubber end of the pencil at A. Keeping this end of the pencil always at A, that is, using A as a pivot, revolve the pencil. Notice that the pencil in the course of its turning will at one time be in the position AC and that it will eventually return to the position AB. Each position of the pencil represents a different angle and the rotation therefore represents all angles. It is evident from this exercise that *the size of an angle depends on the amount of turning and not on the length of the pencil.*

5. Make a general conclusion about the size of an angle that will include both ex. 3 and ex. 4.

6. Show how to transfer an angle BAC to the line DE so that DE will be one side and O (any point in DE) the vertex. Use a strip of paper, MNW, one edge of which, MW, is a straight line.



7. Draw any angle. Transfer it to a point in a given line.

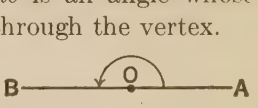
8. Draw two angles of different size on your paper. Then draw (*a*) an angle equal to the sum of the two angles; (*b*) an angle equal to the difference between the two angles.

29. Generating an Angle. We have seen that the size of an angle depends on the amount of rotation necessary to bring a line from the position of one side of the angle to the position of the other side. The line then *generates* the angle.

When the generating line makes a complete revolution the angle formed is called a *perigon.*

30. Straight Angle. A *straight angle* is an angle whose sides form one straight line extending through the vertex.

Thus, in the drawing $\angle AOB$ is a straight angle.

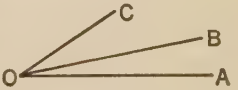


Note that a straight angle is formed when the generating line has made a half-revolution.

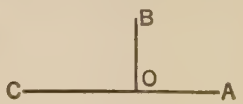
31. PRINCIPLE 12. *All straight angles are equal.* (See § 28.)

32. Adjacent Angles. When two angles have the same vertex and a common side between them, they are called *adjacent angles.*

Thus, in the drawing $\angle AOB$ and $\angle BOC$ are adjacent angles.



33. Right Angle. When one straight line meets another so as to make two equal adjacent angles, each angle is called a *right angle.* A right angle is one half of a straight angle.



Thus, $\angle AOB$ is a right angle.

Note that a right angle is formed when the generating line has made one fourth of a revolution.

34. Perpendicular. When two lines form right angles with each other, each line is said to be *perpendicular* to the other, and the point which the lines have in common is called the *foot* of the perpendicular.

Thus, in the drawing in § 33, BO is perpendicular to AO (written $BO \perp AO$), and O is the foot of the perpendicular.

Exercises — Group 7

1. Make two straight angles AOB and $A'O'B'$. Compare them.

2. Draw a line forming with the edge of the paper a right angle. Fold the paper over on this line and form an equal adjacent angle.

3. Draw two intersecting lines and letter them. Read all the straight angles formed. Mention the pairs of adjacent angles.

4. Fold and crease a sheet of paper. Letter the crease AB. Fold the crease upon itself so that A coincides with B and make a second crease. Unfold to the first crease. What kind of angles are formed? how many? Unfold so that the sheet is opened flat on the desk. What kind of angles are formed by the creases? how many?

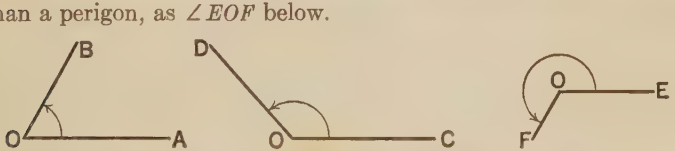
5. Cite examples of the right angle in the classroom.

Oblique Angles

35. An *acute angle* is an angle that is less than a right angle, as $\angle AOB$ below.

36. An *obtuse angle* is an angle that is greater than a right angle but less than a straight angle, as $\angle COD$ below.

A *reflex angle* is an angle greater than a straight angle, but less than a perigon, as $\angle EOF$ below.



37. Acute angles, obtuse angles, and reflex angles are often spoken of as *oblique angles*. Lines which have the position of the sides of any of these angles, that is, lines which are not perpendicular or parallel, are called *oblique lines*.

Exercises — Group 8

1. Draw an acute angle and extend one side to form the adjacent angle. What kind of angle is it? Is this always true?

2. Repeat ex. 1, using an obtuse angle.

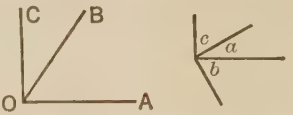
3. What is the quickest way to form two angles whose sum is a straight angle?

4. Could the angles in ex. 3 be both acute? both obtuse? both right? one right and one acute? one acute and one obtuse?

5. Draw an obtuse angle and extend one side to form its adjacent angle. Bisect each of these angles by folding. What kind of angle do the two creases seem to form?

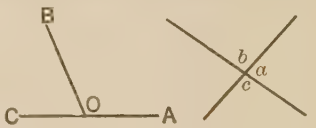
Special Pairs of Oblique Angles

38. Complementary Angles. Two angles are *complementary* when their sum is a right angle. Each angle is called the *complement* of the other.



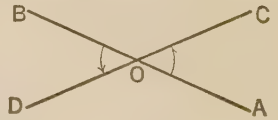
Thus, if $\angle AOC$ is a right angle, $\angle BOC$ is the complement of $\angle AOB$; similarly $\angle a$ is the complement of $\angle b$ and also of $\angle c$.

39. Supplementary Angles. Two angles are *supplementary* when their sum is a straight angle. Each angle is called the *supplement* of the other.



Thus, $\angle AOB$ is the supplement of $\angle BOC$; also $\angle a$ is the supplement of $\angle b$ and also of $\angle c$.

40. Vertical Angles. When two lines cross, the two angles not adjacent are called *opposite* or *vertical angles*.



Thus, $\angle AOC$ and $\angle BOD$ are vertical angles. Name another pair.

Exercises — Group 9

1. What kind of angles are complementary angles? Can both angles be obtuse? Can both be acute? Can one angle be obtuse?

2. Repeat ex. 1 for supplementary angles.

3. Draw an acute angle. By folding as in ex. 4, p. 13, form its complement. What kind of angle is it?

4. Draw an obtuse angle. Form its adjacent supplementary angle. Do this in two different ways. What kind of angle is the supplement in each case? Do the two look equal?

5. Repeat ex. 4 for an acute angle.

6. Draw an acute angle. Represent at its vertex the complement and the supplement of the angle. What is the difference between these two angles? Repeat for different acute angles.

7. Draw an obtuse angle. At its vertex represent the supplement of the angle and call it $\angle a$. By folding, show by what angle the obtuse angle exceeds a right angle. Call this $\angle b$. What is the sum of $\angle a$ and $\angle b$?

8. Draw an obtuse angle. Denote its supplement by $\angle a$ and the difference between the angle and a perigon by $\angle b$. What is the value of $\angle b - \angle a$? Use a drawing.

9. Draw an acute angle. Construct an angle equal to it. (See ex. 6, p. 11.) By folding, represent the complement of each angle. How do these two complements seem to compare in size?

10. Draw an acute angle AOB. By creasing along side OA form the adjacent complementary angle. Call it $\angle a$. Do the same thing, using side OB, and call the angle $\angle b$. Compare $\angle a$ and $\angle b$.

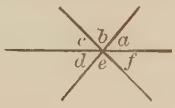
11. Repeat ex. 9 for the supplements.

12. Draw an obtuse angle AOB. Construct its adjacent supplementary angle, first on OA and then on OB. How do these two supplements compare?

13. Can the complement of an angle be greater than the angle? less than the angle? equal to the angle?

14. Repeat ex. 13 for the supplement of an angle.

15. In the drawing point out three different pairs of vertical angles. What angle is the vertical angle of $\angle(d + c)$? What angle is the supplement of $\angle a$? of $\angle(e + f)$? Give two answers to each of the last two questions.



16. What is the sum of $\angle a + \angle b + \angle c$? (See figure p. 15.) Write this in the form of an equation, thus, $\angle a + \angle b + \angle c = ?$ Write five other equations, using each time three consecutive angles.

17. How do the second members of all the equations in ex. 16 compare? Then how should all the first members compare?

18. Using the conclusion of ex. 17, form other equations in which each member consists of three consecutive angles. For example, $\angle a + \angle b + \angle c = \angle b + \angle c + \angle d$. Why?

19. If $\angle a + \angle b + \angle c = \angle b + \angle c + \angle d$, compare $\angle a$ and $\angle d$. Explain. What kind of angles are $\angle a$ and $\angle d$?

20. Show that $\angle b = \angle e$. Show also that $\angle c = \angle f$.

21. From ex. 19 and 20 what conclusion do you make regarding any pair of vertical angles formed by two intersecting lines? Does your conclusion depend on the accuracy of the drawing used?

22. Draw two complementary adjacent angles. Form their corresponding vertical angles. Are they complementary? Why?

23. Repeat ex. 22 for two supplementary adjacent angles.

Measuring Angles

41. When we measure any magnitude, the unit used must be of the same kind as the quantity to be measured. We know that when we measure lines, the unit used is a line. Likewise when we measure angles, the unit used is an angle.

One unit for measuring lines is an arbitrarily fixed length called a *yard*. For greater accuracy in measuring short distances, this unit is in turn divided into a definite number of parts, giving *feet*, *inches*, *half inches*, etc.

Likewise, one unit for measuring angles is determined by dividing a fixed angle (the perigon) into a definite number of equal parts (360).* One of these angles is used as the unit

* The length of the year of 360 days, as worked out by the Babylonian astronomers, is supposed to have furnished the occasion for the division of the perigon into 360 equal parts.

for measuring angles and is called a *degree* (written 1°). For greater accuracy of measurement the degree is divided into 60 equal parts, each called a *minute* (written 1′). For measurements that require still greater accuracy the minute is divided into 60 equal parts, each called a *second* (written 1″).

Exercises — Group 10

1. How many degrees are there in a right angle? in a straight angle? Why?

2. Between what two limits, expressed in degrees, are all acute angles found? all obtuse angles?

3. What kind of angle is each of the following: 37°, 105°, 76°, 147°, 169°, 89°, 15°, 179°, 180°, 205°, 360°, 90°?

4. Read the following angles: 15° 17′, 89° 46′ 57″, 156° 35′ 42″.

5. Find the complement of each of the following angles: 72°, 35°, 48° 24′, 65° 48′, 84° 56′.

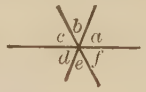
6. Find the supplement of each of the following angles: 80°, 120°, 76° 37′, 98° 57′, 146° 6′, 121° 45′.

7. Express each of the following angles in terms of degrees and fractions of degrees only: 50° 30′, 85° 45′, 125° 20′ 30″.

8. How many degrees are there in one half of a right angle? in one third of a straight angle? in one fifth of a perigon?

9. If two complementary angles are equal, what is the size of each?

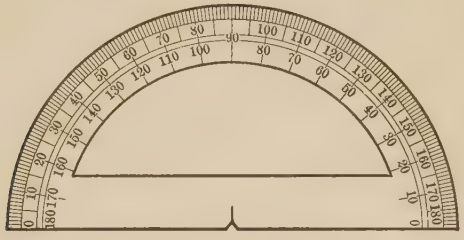
10. If $\angle a = 70°$ and $\angle c = 60°$, find all the other angles.



11. Solve ex. 6, 7, and 8 on p. 15 by use of the equation.

SOLUTION OF EX. 6. If a = the size of the acute angle in degrees
then $90° - a$ = the size of its complement in degrees
and $180° - a$ = the size of its supplement in degrees
Hence the difference $= 180° - a - (90° - a)$
$= 180° - a - 90° + a$
$= 90°$, a right angle

42. The Protractor. The instrument used for measuring angles is called a *protractor*. There are various forms in which protractors are made. One of the common forms is shown in the drawing below.



43. Use of the Protractor. *To measure an angle with a protractor*, place the notch of the protractor at the vertex of the angle and the base edge of the protractor along one side of the angle. The division point on the protractor, which falls on the other side of the angle (produced if necessary), when read from the protractor will indicate the number of degrees in the angle.

To construct an angle of a given size with a protractor, draw a line to be used as one side of the required angle and select a point on this line to be used as the vertex. Place the protractor so that its base edge coincides with the line and its notch with the point. Make a dot on the paper opposite the division point on the protractor which represents the required number of degrees. Remove the protractor and join this point with the vertex. The angle formed is the required angle.

Exercises — Group 11

1. Draw angles of various sizes and measure them with a protractor.

2. Construct the following angles: 40°, 72°, 30°, 90°, 45°, 60°, 32°, 68°, 140°, 120°, 167°, 115°, 180°, 153°, 240°, 315°.

3. Draw the following angles as accurately as possible without the aid of a protractor. Measure the angles and note the errors: 90°, 45°, 30°, 60°, 120°, 180°, 72°, 108°, 40°, 135°, 57°, 22° 30′.

4. Continue this form of exercise until angles of a given size can be drawn, with reasonable accuracy, without the use of a protractor. The eye should be especially trained to estimate angles of 90°, 60°, 30°, 45°, and 180°.

5. Draw an angle. Without the aid of a protractor draw another angle as nearly as possible equal to the first angle. Measure the angle and note your error.

6. Draw an angle. Draw as accurately as possible, without the use of a protractor, the bisector of the angle. Measure each angle and note your error. Repeat this exercise until the bisector can be drawn with reasonable accuracy.

7. At a point in a line draw, without the use of a protractor, a perpendicular to the line. Note your error. Repeat the exercise.

8. From a point outside a line draw free-hand a perpendicular to the line. Note your error. Repeat the exercise.

9. At the end of a line construct, with the aid of a protractor, a right angle. Extend one side to form the adjacent angle. Measure this angle with the protractor. Compare the two angles.

10. Draw an acute angle and draw its complement determined in two different ways.

11. Draw an acute angle and draw its supplement determined in two different ways.

12. Draw an obtuse angle and draw its supplement determined in two different ways.

13. What is the complement of each of the following angles: 30°, 45°, 60°, 72°, 18°? Construct each pair with a protractor.

14. What is the supplement of each of the following angles: 90°, 120°, 95°, 54°, 163°, 175°? Construct each pair of angles with a protractor.

Exercises — Group 12

NUMERICAL

1. Two angles are complements and one is twice the other. What are the angles? Construct them.

2. Two angles are complements and one is 10° larger than the other. What are the angles? Construct them.

3. Two angles are complements and one is 18° less than the other. What are the angles?

4. Two angles are complements and one lacks 8° of being twice the other. What are the angles? Construct them.

5. Two angles are complements and one is 19° more than three times the other. What are the angles?

6. Three angles whose sum is a right angle are in the ratio 1 : 2 : 3. Find the angles and construct them, using the protractor.

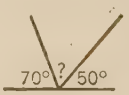
7. Two angles are supplementary and one is five times the other. Find the angles and construct them.

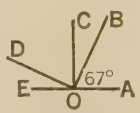
8. Two angles are supplementary and one is 40° less than the other. Find the angles and construct them.

9. The complement of a certain angle is 220° less than twice its supplement. Find the angle and construct it.

10. The supplement of a certain angle is 20° more than twice its complement. Find the angle and construct it.

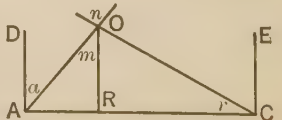
11. In the second drawing at the right, $CO \perp AO$ and $BO \perp DO$. Find all the angles in both figures.





12. The complement and the supplement of a certain angle when added equal 170°. Find the angle and construct it.

13. In the drawing DA, OR, and EC are each $\perp AC$ and $\angle a = 40°$, $\angle m = 40°$, $\angle n = 100°$, $\angle r = 30°$. Find all the other angles of the drawing.

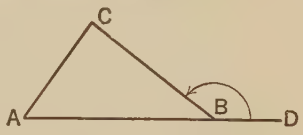


14. What angles do the hands of a clock make with each other in the following instances: 3 o'clock, 2.00, 5.00, 11.00, 5.12, 1.24, 11.36?

TRIANGLES

44. A figure bounded by three straight lines is called a *triangle* (symbol $\triangle$). The three lines are called the *sides*, and their intersections, the *vertices*, of the triangle. Every triangle has *six parts:* three sides and three angles.

Thus, in the drawing at the right, AB, BC, and CA are the sides, A, B, and C the vertices, and $\angle A$, $\angle ABC$, and $\angle C$ the angles of the triangle.



45. The side upon which a triangle is supposed to stand is usually called its *base*. In general, however, any side may be considered the base. The angle opposite the base is the *vertex angle* and the vertex of this angle is called the *vertex* of the triangle.

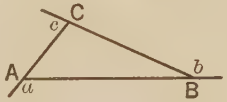
Thus, in the above drawing, AB is the base, $\angle C$ the vertex angle, and C the vertex of triangle ABC.

46. An angle formed by one side of a triangle and the adjacent side produced is called an *exterior angle of the triangle*.

In the above drawing, $\angle DBC$ is an exterior angle. In this case $\angle A$ and $\angle C$ are referred to as the *two opposite interior angles*.

47. In forming a set of exterior angles for a triangle, we *produce the sides in order*.

Thus, in the drawing at the right, $\angle a$, $\angle b$, and $\angle c$ constitute a set of exterior angles of $\triangle ABC$.



Exercises — Group 13

1. Copy the drawing of § 47 and form another set of exterior angles.

2. Draw a triangle and cut it out with a pair of scissors. Cut off the angles and place them side by side at one vertex. Repeat with other triangles. What seems to be the sum?

3. Draw any triangle, measure each angle, and find the sum. Repeat with other triangles. What seems to be the sum?

4. Draw a large triangle ABC. Place your pencil along AB, *noting the direction in which the pencil points.* Revolve the pencil through angle B, then through angle C, and then through angle A, using in turn vertices B, C, and A as pivots. The pencil should now fall along AB again. *In what direction does the pencil now point?* Then through what angle did it rotate? Repeat for other triangles. Hence, what seems to be the sum of the three angles?

48. From ex. 2 to 4 we conclude that:

The sum of the three angles of any triangle is equal to one straight angle.

Ex. 1. Which of the following combinations are possible for the three angles of a triangle? Give reasons in each case.

(*a*) Three acute angles
(*b*) Three obtuse angles
(*c*) Three right angles
(*d*) Two acute angles, one obtuse angle
(*e*) Two acute angles, one right angle
(*f*) Two obtuse angles, one acute angle
(*g*) Two right angles, one acute angle

***Ex. 2.** Draw a triangle ABC. Produce AB to D, forming the exterior angle DBC. Measure angle DBC, angle A, and angle C. Compare angle DBC with the sum of angle A and angle C. Repeat for other vertices and other triangles. Form a conclusion.

Ex. 3. An exterior angle of a triangle is 148° and one of the opposite interior angles is 67°. What are the three angles of the triangle? What are the other two exterior angles which with angle 148° form a set? Find each of the three angles in the other set of exterior angles. Make careful drawings to illustrate this exercise.

Ex. 4. The three angles of a triangle are in the ratio 3 : 5 : 7. What are the angles?

Ex. 5. Draw any triangle. Compare the lengths of the sides. (Use compasses.) Number them 1, 2, 3, according to their lengths, 1 being the longest. Likewise compare and number the angles 1, 2, 3. Repeat for different triangles. Make a conclusion.

CLASSIFICATION OF TRIANGLES WITH REFERENCE TO THEIR SIDES

49. Scalene Triangle. A triangle no two sides of which are equal is called a *scalene triangle.*

50. Isosceles Triangle. A triangle two sides of which are equal is called an *isosceles triangle.*

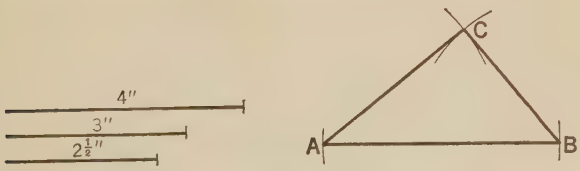
The equal sides of an isosceles triangle are usually referred to as the *legs* of the triangle, the unequal side as the *base,* the angles at the base as the *base angles,* and the angle opposite the base as the *vertex angle.*

51. Equilateral Triangle. A triangle all three sides of which are equal is called an *equilateral triangle.*

These three classes of triangles are illustrated below.

Exercises — Group 14

1. Construct a triangle whose sides are 4″, 3″, $2\frac{1}{2}$″.



METHOD. Draw a line 4″ long. With the points of the compasses 3″ apart and one point on one end of the line draw an arc of a circle above the line. Repeat the process at the other end of the line with the points of the compasses $2\frac{1}{2}$″ apart. Connect the intersection of the two arcs with the ends of the given line. The triangle formed is the required triangle.

2. Construct a triangle whose sides are: 3″, $1\frac{1}{2}$″, $2\frac{3}{8}$″; 2″, $2\frac{3}{4}$″, $3\frac{1}{2}$″.

3. Construct to scale (1″ representing 4′) a triangle whose sides are 13′, 11′, 6′.

4. Construct to scale a triangle whose sides are 6′, 8′, 10′.

5. Can three lines, selected at random, be used as sides of a triangle? Make drawings and discuss your answer.

6. Construct a triangle whose sides are: 2″, 3″, 3″; 4″, $3\frac{1}{4}$″, $3\frac{1}{4}$″; $3\frac{3}{4}$″, 5″, 5″.

7. Construct to scale a triangle whose sides are 12′, 18′, 18′.

8. Measure all the angles in an isosceles triangle. Check the sum. Does there seem to be any relation between the base angles? Repeat, using different isosceles triangles, and make a conclusion.

9. Verify your conclusion in ex. 8 by folding an isosceles triangle over on its base so that the two ends of the base coincide.

10. Can the base angles of an isosceles triangle be obtuse? right? Explain, using drawings.

11. Can the vertex angle of an isosceles triangle be acute? right? obtuse? Explain, using drawings.

12. What relation exists between the base and either of the equal sides of an isosceles triangle when the vertex angle is obtuse? Is this consistent with the observations made in ex. 5, p. 22?

13. Construct an equilateral triangle, each of whose sides is 3″; $2\frac{1}{2}$″; 4″.

14. Measure each angle in each triangle of ex. 13. How do they seem to compare? What seems to be the size of each angle?

15. Draw a line 3″ long. At one end construct an angle of 40°; at the other end, an angle of 60°. Extend the sides to form a triangle. What should be the size of the third angle? Measure and check. Determine from the angles whether this should be a scalene, an isosceles, or an equilateral triangle.

16. Repeat ex. 15 for the following triangles: 4″, 35°, 65°; $2\frac{1}{2}$″, 70°, 40°; $2\frac{3}{8}$″, 110°, 50°; $3\frac{1}{2}$″, 60°, 60°; 3″, 64°, 58°; $2\frac{3}{4}$″, 58°, 32°.

Classification of Triangles with Reference to Their Angles

52. Acute Triangle. A triangle each of whose angles is an acute angle is called an *acute triangle.*

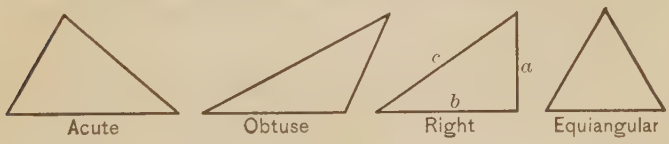
53. Obtuse Triangle. A triangle one of whose angles is an obtuse angle is called an *obtuse triangle.*

54. Right Triangle. A triangle one of whose angles is a right angle is called a *right triangle.* The side opposite the right angle is called the *hypotenuse* and the other two sides are called the *legs* of the triangle.

Thus, in the third figure below, c is the hypotenuse, and a and b are the legs of the triangle.

55. Equiangular Triangle. A triangle all of whose angles are equal is called an *equiangular triangle.*

These four classes of triangles are illustrated below.



Exercises — Group 15

1. Can any conclusion be made concerning each of the two remaining angles in an obtuse triangle? in a right triangle?

2. Can any conclusion be made at once from the definition as to the size of each angle in an equiangular triangle?

3. Draw any obtuse angle. Select a point in each arm and join the two points, thereby forming a triangle. What kind of angles should the two remaining angles be? How should the two points be selected if the two remaining angles are to be equal?

4. Repeat ex. 3, using an acute angle.

5. Construct a right angle with the protractor. Select any point in each arm of the angle and join the two points, thereby forming a triangle. What kind of angles will the two remaining angles be? How should the points be selected if the two remaining angles are to be equal? In this case what would be the size of each angle?

6. Compare the length of the side opposite the obtuse angle in any obtuse triangle with the other two sides. (See ex. 5, p. 22.)

7. Compare the length of the side opposite the right angle in a right triangle with the other two sides. (See ex. 5, p. 22.)

8. On any line as a base construct a triangle whose angles are: 40°, 80°, 60°; 72°, 48°, 60°; 28°, 67°, 85°. Was it necessary to give all three angles in each case? Explain.

9. Could any three angles, selected at random, be used in ex. 8? Explain.

10. On any given line as a base construct a triangle whose two base angles are: 120°, 40°; 130°, 20°; 98°, 47°; 60°, 90°; 52°, 52°; 60°, 60°. Check the third angle in each case.

11. Could any two angles be used in ex. 10? Could any two acute angles be used? Explain.

12. Could the following set of angles be used for the angles of a triangle: 59°, 37°, 83°; 43°, 97°, 42°; 63°, 52°, 65°?

13. What is another name for the equiangular triangle?

14. From the angles 45°, 23°, 72°, 58°, 23°, 45°, 72° select pairs that might be used in each of the following triangles:

(*a*) An acute scalene triangle (*c*) An acute isosceles triangle
(*b*) An obtuse scalene triangle (*d*) An obtuse isosceles triangle
(*e*) A right isosceles triangle

15. Can any triangle be constructed in which all the exterior angles will be (*a*) acute? (*b*) obtuse? Illustrate.

16. When, after 3 o'clock, will the hands of a clock form an angle of 108°?

17. What angles do the hands of a clock form at 8 o'clock? at 7.48 o'clock? at 6 o'clock? at $10\frac{10}{11}$ minutes after 5 o'clock?

POLYGONS

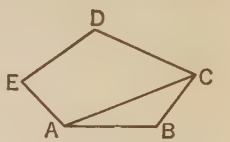
56. We have been studying the plane figure that has three sides. A plane figure, however, may have 4, 5, 6 . . . sides. The term used to include them all is *polygon*. (Compare the term *polynomial* in Algebra.)

57. A *polygon* is a plane figure bounded by straight lines. The parts are defined as in the triangle. (See §§ 44 to 47.) Such a figure is often called a *plane rectilinear figure*.

Two angles at the end of any side are called *consecutive angles* and are said to *include* that side. Two sides that meet at any vertex are called *consecutive* or *adjacent sides* and are said to *include* the angle at that vertex.

A *diagonal of a polygon* is a line joining two vertices that are not consecutive.

Thus, in the drawing $ABCDE$ is a polygon, AB, BC, . . . are the sides, A, B, C, . . . are the vertices, AC is a diagonal.



Exercises — Group 16

1. Draw any polygon of 4 sides. How does the number of angles compare with the number of sides? Measure each angle and find the sum.

2. Repeat ex. 1 for polygons having 5, 6, 8 sides. Fill in the following table, in which n represents the number of sides.

n	No. of Angles	Sum of the Angles	No. of St. Angles in the Sum
3			
4			
5			
6			
8			

3. Does there seem to be any relation between the number of straight angles in any sum and the number of sides in that polygon?

4. How many diagonals can be drawn from one vertex in a polygon of 4 sides? of 5 sides? 6 sides? 8 sides? 10 sides? n sides?

5. In ex. 4 how does the number of triangles formed compare with the number of diagonals in each case?

6. If all the vertices are joined to some point inside the polygon, how many triangles are formed? Consider the general case in which the polygon has n sides.

7. How many triangles are formed (*a*) if the point is in one side, but is not a vertex? (*b*) if the point is a vertex?

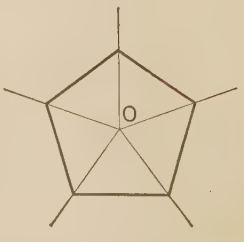
8. Using all vertices, how many diagonals can be drawn in a polygon of 4 sides? of 5 sides? of 6 sides?

9. Draw any polygon of 4 sides, form a set of exterior angles (see § 47), measure, and find their sum.

10. Repeat ex. 4 for polygons having 5, 6, 7 sides. What is the sum in each case?

58. Regular Polygons. If a polygon has all its angles equal and all its sides equal, it is called a *regular polygon.*

A simple method of constructing a regular polygon of n sides (when n is an exact divisor of 360) is to divide a perigon (360°) into n equal parts, with the aid of a protractor, select a point on each arm the same distance from the common vertex, and join these points in order. In the drawing, $n = 5$, hence the resulting polygon is a regular polygon of 5 sides.



59. The common vertex for all the angles is called the *center* of the polygon. In the drawing O is the center.

Exercises — Group 17

1. Construct a regular polygon of 5 sides, as suggested in § 58. Test the sides and the angles.

2. Is there any relation between one of the angles of the polygon and one of the angles at the center? Use a protractor.

3. Repeat ex. 1 and ex. 2 for regular polygons of 4, 6, 8 sides.

4. Make a general conclusion about the relation between an angle of any regular polygon and one of the angles at the center.

5. If one of the angles of a regular polygon is 135°, what is one of the angles at the center? Hence how many sides has the polygon?

6. How many sides has a regular polygon when each angle is 108°? 144°? 120°? 150°? 140°? 90°? $128\frac{4}{7}$°?

7. Can any angle be selected at random as one of the angles of some regular polygon? Explain.

8. Construct a regular polygon of 6 sides. Starting at any vertex connect alternate vertices, that is, every other one. What kind of figure is formed? Test with a ruler and a protractor.

9. Repeat ex. 8 for a regular polygon of 8 sides.

10. What is the size of each angle of a regular polygon if the number of sides is 4? 5? 6? 7? 8? 9? Reverse the order of work in ex. 5 in solving this exercise.

11. Using a line 2″ long for one side, construct a regular polygon of 6 sides by successive construction of the angle of such a polygon as found in ex. 10. Check the last two angles by using the protractor.

12. Using the polygon constructed in ex. 11, draw all the diagonals from one vertex. By measurement determine what kind of triangles are formed.

13. Using the method of ex. 11, construct (*a*) a regular polygon of 4 sides on a line 2″ long; (*b*) a regular polygon of 8 sides on a line 1″ long.

THE CIRCLE

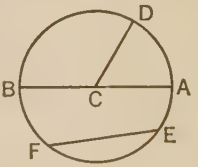
60. A *circle* is a closed curve in a plane all of whose points are equally distant from a fixed point in the plane called the *center*.

Circles are usually named from their center.
Thus, we have circle O, in which the center is O.

61. The *circumference* is the length of the circle. A *semicircumference* is one half the length of the circle.

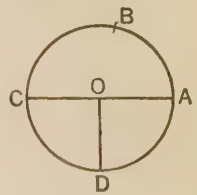
Thus, in the drawing of § 62, the circumference is the length of the curve $ADBFEA$, and the semicircumference is the length of the curve ADB.

62. A line through the center with its ends in the circle is called a *diameter*. A line joining the center with any point in the circle is called a *radius* (plural, *radii*). Any line whose ends are in the circle is called a *chord*.



Thus, in the circle C, AB is a diameter, CD is a radius, and EF is a chord.

63. Any portion of a circle is called an *arc*. The arc which represents half a circle is called a *semicircle*. The arc which represents a quarter of a circle is called a *quadrant*.



Thus, in circle O, AB is an arc (written $\overset{\frown}{AB}$ and read "arc AB"), $\overset{\frown}{ABC}$ is a semicircle, and $\overset{\frown}{AD}$ is a quadrant.

64. The circle is the only curved line studied in Plane Geometry. Its frequent use in various fundamental constructions and its numerous applications outside of geometry amply justify its study.

Compasses are the instruments used for drawing circles.

Exercises — Group 18

1. Is every diameter of a circle a chord? Is every chord a diameter?

2. If two radii or two diameters are drawn in a circle, how many points do they always have in common? Explain.

3. How do all radii in the same circle compare? all diameters?

4. Show that a radius is one half a diameter.

5. Show that any two diameters bisect each other.

6. Draw a circle and in it a diameter. Select any point in either semicircle and join it with the ends of the diameter. What kind of angle is formed between the two chords? Is this always true?

7. By drawing around a coin or some circular object, construct a circle. Draw in it any chord and at one end draw a second chord at right angles to the first. Join by a line the ends of the two chords and find by measurement, as accurately as possible, the mid-point of this third chord. Is this the center of the circle?

8. Draw a circle and in it two diameters at right angles to each other. Join the ends of these diameters and show by measurement that the polygon formed is regular. What are the arcs?

9. Draw a circle. With the compasses opened a distance equal to the radius, and starting at any point on the circle, lay off this distance successively around the circle. Join the points in order and determine by measurement whether the polygon formed is regular.

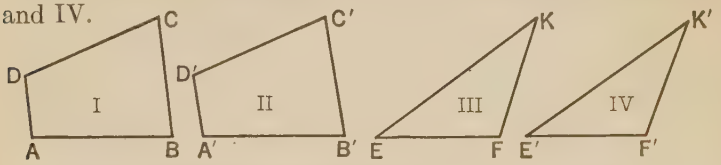
10. In ex. 9 is the length of the arc or of the chord between successive points equal to the radius? Explain.

11. Draw a line AB $1\frac{1}{2}''$ long and draw the perpendicular bisector of AB. Use a ruler and a protractor. Show that any point on this perpendicular could be used as the center of a circle passing through both A and B.

12. In ex. 11, how many circles can be drawn passing through A and B? What is the radius of the smallest of these circles? In this case what is AB called? In all other cases, what is AB called?

CONGRUENCE

65. Congruent Figures. Two plane figures are said to be *congruent* (symbol $\cong$) if they can be made to coincide. This means that if two such figures are cut from paper, it is always possible to make one exactly match the other. This process is called *superposition*. The angles and sides which would match are called *corresponding* or *homologous parts*. In the following drawings, I and II are congruent; also III and IV.



Name the corresponding parts in I and II; in III and IV.

66. The fundamental test as to the congruence of two figures is: *Can they be made to coincide throughout?*

Superposition is usually an imaginary process, as the transfer does not in general actually take place.

67. It is evident from § 66 that if two plane figures are congruent, they must have the same *shape* and *size*. However, in comparing two geometric magnitudes where the question of shape does not enter, as, for example, two lines or two angles, the word *equal* is usually used in place of *congruent*. (See §§ 14 and 28.)

Exercises — Group 19

1. How do corresponding parts in congruent figures compare?

2. Is it necessary when two figures are congruent for all the angles and all the sides to be equal, or merely for corresponding pairs to be equal?

3. Draw any line. Construct three different triangles, each having the given line for its base. Are the triangles necessarily congruent? Cut the triangles out and test.

4. Construct three different triangles, each having a given line for its base and an angle of 40° at the same end of the base. Are these triangles necessarily congruent? Cut the triangles out and test.

5. Construct three different triangles, each having a given line for the base, an angle of 50° at one end, and an angle of 60° at the other end. Are these triangles necessarily congruent? Cut the triangles out and test.

68. From the above exercises it is evident that two triangles must have *certain parts* equal respectively* if the triangles are to be congruent. Another way of stating this is to say that certain of the six parts of any triangle *determine* the triangle, meaning that two or more triangles constructed with these parts equal can be made to coincide.

Ex. 1. Construct any two equilateral triangles, using two unequal lines for bases. Are the three angles of the one triangle equal respectively to the three angles of the other triangle? Are the triangles congruent? Cut the triangles out and test.

Ex. 2. Construct a triangle on a 3-inch base with angles of 45° and 75° at the ends. Construct another triangle on a 4-inch base with the same angles at the ends. Are the angles equal respectively? Are the triangles congruent?

Ex. 3. Does the fact that the three angles of one triangle are equal respectively to the three angles of another triangle necessarily make the two triangles congruent?

From ex. 3 the truth of § 68 is made more apparent. That is, it is evident that the parts that make two triangles congruent cannot be selected at random; certain parts must be selected.

Ex. 4. Are two circles equal (see § 67) if they have equal radii?

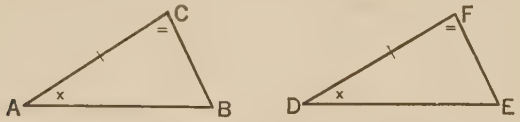
Ex. 5. Are two equilateral triangles equal if they are constructed on equal bases?

* Parts of two figures are said to be considered *respectively* when they are considered in the order named. For example, in figures III and IV, § 65, $\angle E$, EF, and $\angle F$ are equal respectively to $\angle E'$, $E'F'$, and $\angle F'$.

69. Pairs of lines or angles may frequently be proved equal by showing that they are corresponding parts of two *congruent* triangles. (See § 65.)

70. In two congruent triangles parts that are known to be equal are always corresponding parts. Of the other parts, a pair lying opposite to any two equal parts will themselves be corresponding parts and therefore equal.

To illustrate, suppose that $\triangle ABC \cong \triangle DEF$ and that $\angle A = \angle D$, $AC = DF$, and $\angle C = \angle F$. Then we know at once that $\angle A$ and $\angle D$ are corresponding parts. It follows that lines BC and EF, lying opposite this pair of equal parts, are also corresponding parts and therefore are equal.

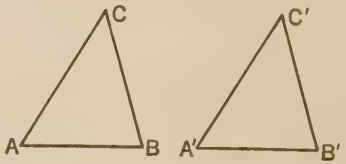


Exercises — Group 20

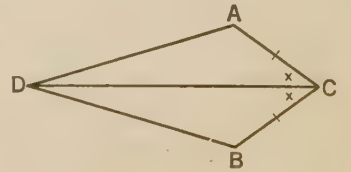
1. In the figure of § 70, mention all the other corresponding parts and tell why they are corresponding parts.

2. If $\triangle RST \cong \triangle MNP$ and $RT = MP$, $TS = PN$, and $\angle T = \angle P$, mention the corresponding parts; the equal parts.

3. In the figure $\triangle ABC \cong \triangle A'B'C'$ and it is given that $\angle A = \angle A'$, $AB = A'B'$, $\angle B = \angle B'$. Mention the six pairs of corresponding parts in the two triangles and state in each case why they are corresponding parts. What parts are equal?

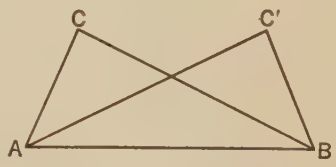


4. In the figure $\triangle ADC \cong \triangle BDC$ and $AC = BC$, $\angle ACD = \angle BCD$, and $DC = DC$. Mention the six pairs of corresponding parts in the two triangles and state why they are corresponding parts.



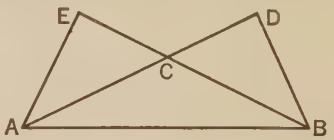
5. In the figure of ex. 3, assume that $AC = A'C'$, $\angle C = \angle C'$, and $BC = B'C'$. Mention the six pairs of corresponding parts in the two triangles.

6. In the figure $\triangle ABC \cong \triangle ABC'$ and it is given that $\angle CAB = \angle C'BA$ and $\angle ABC = \angle BAC'$. Mention the corresponding parts in the two triangles.



NOTE. In studying these "overlapping" triangles it will be found helpful to make large pasteboard models, lettering them and placing them as in the figure.

7. In the figure $\triangle ABD \cong \triangle BAE$, and it is given that $AD = BE$ and $\angle DAB = \angle EBA$. Mention all the corresponding parts in the two triangles.



8. In the figure of ex. 7, suppose that $\triangle AEC \cong \triangle BDC$, and it is given that $AC = BC$ and $EC = DC$. Mention the various pairs of corresponding parts in the two triangles.

CONSTRUCTIONS

71. Some of the most common problems of construction are taken up here. The ruler, compasses, and protractor may be used to test the accuracy of the work. The proofs that the methods used are correct must, however, be deferred. Since the truth of conclusions arrived at through a course of reasoning should not depend on the accuracy of the drawing used, it does not destroy the validity of a proof to assume the construction of certain lines and angles, *provided* that it is shown that such lines and angles actually exist.

To insure greater accuracy in the following problems, which are purely construction problems, all lines in the drawings are made solid — auxiliary lines are light, given lines are medium, and required lines are heavy.

Problem I

72. *To bisect a given line.*

Given: Line AB.
Required: To bisect AB.

Method:

1. With A as center and a radius greater than one half AB, describe arcs above and below AB.
2. With B as center and the *same* radius, describe arcs intersecting the other two at C and D respectively.
3. Draw the st. line through C and D.
4. Point P, where CD crosses AB, is the required point.

The correctness of this method may be established by § 119.

Discussion:

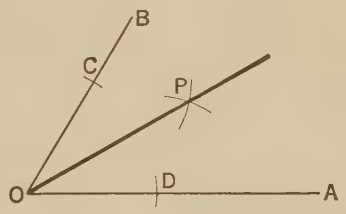
1. Test this bisection with compasses.
2. With a protractor measure $\angle CPA$. Line CD is called the *perpendicular bisector* of line AB.
3. Is AB the perpendicular bisector of CD?
4. Why must the radius be greater than one half AB?
5. Must the same radius be used in obtaining both C and D?
6. Could C and D both be on the same side of AB?

Ex. Construct the three perpendicular bisectors of a triangle. Repeat for several triangles.

Note. If your construction is correct, these perpendiculars will pass through the same point.

PROBLEM II

73. *To bisect a given angle.*



Given: $\angle AOB$.

Required: To bisect $\angle AOB$.

Method:

1. With O as center and any radius, strike arcs on sides OA and OB, cutting them at D and C respectively.
2. With C as center and a radius greater than one half the distance CD, describe an arc within the angle.
3. With D as center and the *same* radius, describe an arc intersecting the last arc at P.
4. Draw the st. line through O and P.
5. Then OP is the required bisector.

The correctness of this method may be established by § 119.

Discussion:

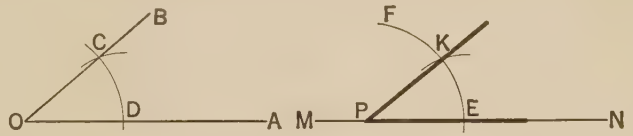
1. Test the bisection by measuring $\angle AOP$ and $\angle POB$.
2. If $\angle AOB$ were a st. $\angle$, how might OP be described?
3. Make a drawing for the case where $\angle AOB$ is obtuse.

Ex. Construct the bisectors of the three angles of a triangle. Repeat for several triangles.

NOTE. If your construction is correct, these bisectors will pass through the same point.

Problem III

74. *At a given point in a given line to construct an angle equal to a given angle.*



Given: $\angle AOB$, line MN, and a point P in line MN.

Required: To construct an angle at P equal to $\angle AOB$ and having NP one arm of the angle.

Method:

1. With O as center and any radius, draw arc DC.
2. With P as center and the *same* radius, draw arc EF.
3. With E as center and distance DC as radius, draw an arc cutting arc EF at K.
4. Draw the line through P and K.
5. Then $\angle EPK$ is the required angle.

The correctness of this method may be established by § 119.

Discussion:

Test the accuracy of the work by measuring the two angles.

Ex. 1. Work Problem III when the given angle is obtuse.

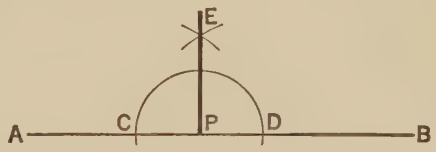
Ex. 2. Could this method be used if the given angle were a right angle? a straight angle? Illustrate.

Ex. 3. Draw an isosceles triangle and verify the fact that the base angles are equal by "transferring" one of the angles to the other by Problem III.

Ex. 4. Draw a polygon of 4 sides. Construct another polygon whose sides and angles shall be equal respectively to the sides and angles of the given polygon.

Problem IV

75. *To draw a perpendicular to a line at a point in the line.*



Given: Line AB and a point in line AB, as P.

Required: At P to draw a line perpendicular to AB.

Method:

1. With P as center and any radius, describe arc CD.
2. With C as center and any radius greater than one half CD, describe an arc above line AB.
3. With B as center and the *same* radius, describe an arc intersecting this last arc at E.
4. Draw the line through E and P.
5. Then EP is the required perpendicular.

The correctness of this method may be established by § 119.

Discussion:

1. Test the work by measuring $\angle BPE$ and $\angle APE$.
2. Could point E have fallen inside arc CD? below AB?
3. Is EP the $\perp$ bisector of AB? of CD?
4. Compare Problem IV with Problem I.

Ex. 1. Show how a perpendicular could be drawn to a line at one end of the line by using Problem IV and then Problem III.

Ex. 2. Draw an equilateral triangle and bisect one of the angles by § 73. Extend this line to meet the opposite side. Verify by Problem IV the fact that the bisector is perpendicular to that side.

Problem V

76. *To draw a perpendicular to a given line from a point outside the line.*

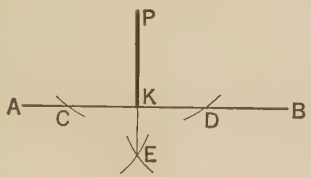


Fig. 1

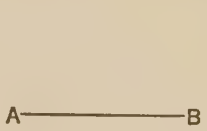


Fig. 2

Given: Line AB and any point P not in line AB.

Required: To draw a perpendicular from P to line AB.

Method:

1. With P as center and the distance from P to any point below AB as radius, draw arcs cutting AB at C and D. (Fig. 1.)
2. With C as center and any radius greater than one half CD, describe an arc below AB.
3. With D as center and the *same* radius, describe an arc intersecting the last arc in E.
4. Draw the line through P and E. Let it cross AB at K.
5. Then PK is the required line.

The correctness of this method may be established by § 119.

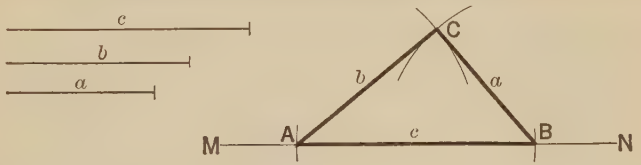
Discussion:

1. Test the accuracy of the work. How is this done?
2. Will K be the center of AB? of CD? Explain.
3. Could E have fallen above line AB?
4. How should you proceed if the parts were located as in Fig. 2?

Sug. Extend line AB.

Problem VI

77. *To construct a triangle when the three sides are given.*



Given: Lines a, b, and c.

Required: To construct a triangle whose sides are a, b, and c.

Method:

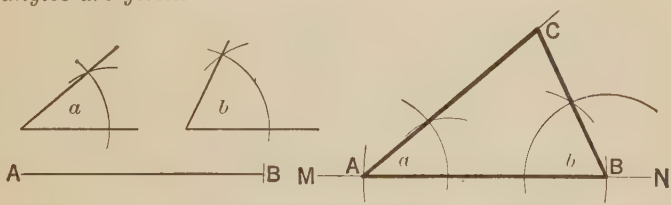
See ex. 1, p. 23.

Discussion:

1. Could a, b, and c be any three lines? Explain.
2. If they were all equal, could any length be used?

Problem VII

78. *To construct a triangle when one side and the adjacent angles are given.*



Given: Line AB and $\measuredangle$ a and b.

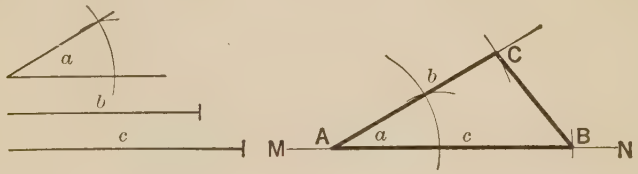
Required: To construct a triangle in which AB is one side and a and b are the adjacent angles.

Method:

Left to the pupil.

Problem VIII

79. *To construct a triangle when two sides and the included angle are given.*



Given: $\angle a$ and lines b and c.

Required: To construct a triangle in which b and c are two sides and $\angle a$ is the included angle.

Method:

Left to the pupil.

Discussion:

1. Could $\angle a$ be taken as any angle?
2. Could b and c be taken any lengths?

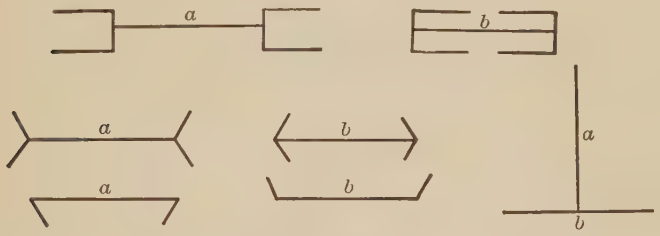
80. The reliability of nearly all discoveries* and constructions made so far has depended solely on the accuracy of the instruments used in measuring and on the accuracy of the eye in using these instruments. Neither the instruments nor the eye is infallible. We often find errors in work due solely to one or both of these factors. For this reason, while we may feel reasonably sure that the discoveries made on previous pages are true, we are never absolutely certain of them because of this possibility of error.

* For discoveries that did not depend on the accuracy of the drawing, see § 12 and ex. 21, 22, p. 16.

As further evidence that it is unsafe to trust the eye to compare magnitudes, consider the following exercises.

Exercises — Group 21

1. In each of the drawings below tell which is longer, a or b. Test with compasses.



2. In Fig. 1 below is CD the prolongation of AB? Apply your ruler to CD and test.

In Fig. 2 which lines are the prolongation of each other?

Fig. 1

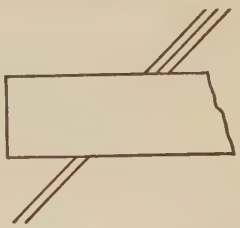


Fig. 2

3. In Fig. 1 below are the two heavy lines straight lines? are they parallel? Test.

In Fig. 2 is a congruent to b? Test by tracing each on thin paper and superposing a on b.

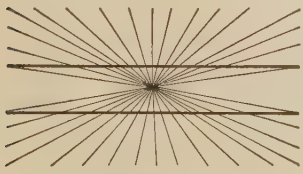


Fig. 1

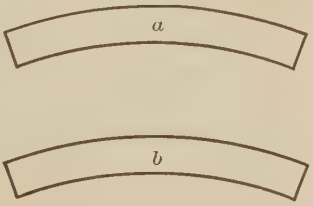


Fig. 2

81. It is necessary, if our discoveries are to be dependable, that we reach our conclusions by a method independent of the accuracy of our instruments or of our eyes. This is attempted in a course in Demonstrative Geometry, where discoveries are based entirely on reasoning instead of on observation. As a beginning for such a course, a certain set of foundation facts in the form of definitions, axioms, etc. is agreed on and accepted as true without question. All discoveries made hereafter are shown to follow inevitably from these accepted facts.

82. Below is given a list of general facts which are accepted as true without question. They constitute a part of the foundation laid down for the course in Demonstrative Geometry. The truth of each statement can readily be tested by illustrations outside of geometry.

1. *Quantities that are equal to the same quantity or to equal quantities are equal to each other.* (§ 18)

2. *If equals are added to equals, the sums are equal.* (§ 21)

3. *If equals are subtracted from equals, the remainders are equal.* (§ 22)

4. *Doubles of equals are equal. In general, if equals are multiplied by equals, the products are equal.* (§ 23)

5. *Halves of equals are equal. In general, if equals are divided by equals, the quotients are equal.* (§ 24)

It is to be noted here that the divisor must not be zero.

6. *The whole is equal to the sum of all its parts.* (§ 19)

7. *The whole is greater than any of its parts.* (§ 20)

8. *A quantity may be substituted for its equal in any process.*

9. *If the first of three quantities is greater than the second quantity, and the second is greater than the third, then the first is greater than the third.* (§ 25)

83. The statements in § 82 are called "axioms." An *axiom* is a general statement admitted without proof to be true.

Other axioms will be added as they are needed.

84. Below are given certain facts peculiar to geometry, which are accepted without proof. These also constitute a part of the foundation for the course in Demonstrative Geometry. The pupil should test each statement to feel convinced of its reasonableness.

1. *Through two given points one and only one straight line can be drawn.* (§ 10)

2. *A straight line is the shortest line that can be drawn between two points.* (§ 9)

3. *A straight line can be extended indefinitely or it may be limited at any point.* (§ 13)

4. *A circle can be described about any point as center and with any given radius.*

5. *A geometric figure may be freely moved in space without any change in form or size.*

85. The above statements are called "postulates." A *postulate* is a geometric statement admitted without proof to be true.

Other postulates will be added as they are needed.

86. From these accepted facts, that is, definitions, axioms, and postulates, certain other facts follow inevitably. A few of the simpler ones will now be treated informally.

87. *Two straight lines can intersect at but one point.* (§ 12)

Ex. Show by drawings that this statement is not generally true of curved lines.

88. *A line can be divided into two equal parts by but one point.*

1. Why has every line *at least one* bisection point? (See § 72.)
2. If P and P' were both bisection points of AB, what lines, beginning at A, would be equal, by Ax. 5?
3. Show that Ax. 7 would be violated. What conclusion follows?

89. *A given angle can be divided into two equal parts by but one line.*

(*a*) Show that every angle has one such bisector. (See § 73.)
(*b*) Show that no angle has two such bisectors.
Use an outline similar to the one in § 88.

90. *All straight angles are equal.*

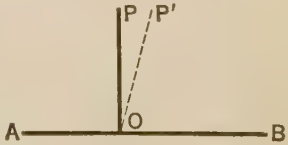
See § 28.

91. *All right angles are equal.*

1. What kind of ∡ are known to be equal?
2. Compare a rt. ∠ with one of these ∡.
3. What axiom, then, makes all right ∡ equal?

Complete the above drawing to show the ∡ referred to in step 1.

92. *Only one perpendicular can be drawn to a line at a point in the line.*



1. Why can *at least one* such ⊥ be drawn? (See § 75.)
2. Suppose OP and OP' are both ⊥ AB. Mention four ∡ that would be equal by § 91.
3. What axiom would be violated? What conclusion follows?
4. Does § 92 at once follow from § 89? Explain.

93. *Complements of equal angles or of the same angle are equal.*

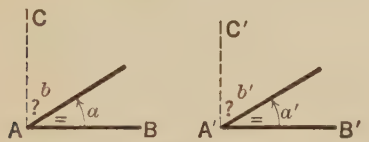


Fig. 1

Fig. 2

Fig. 1. 1. What two pairs of $\measuredangle$ are known to be equal? Why?
2. How is the required pair obtained from these?
3. What axiom therefore makes this pair of $\measuredangle$ equal?

Fig. 2. 1. $\measuredangle$ LMK and NMY are rt. $\measuredangle$. Show that $\angle s = \angle t$.
2. How are $\measuredangle$ s and t obtained from the rt. $\measuredangle$? Then why are they equal?

Put proofs in equation form, using the letters of the drawing.

94. *Supplements of equal angles or of the same angle are equal.*

Fig. 1

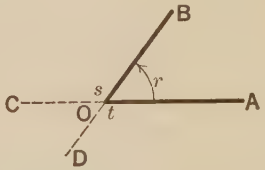


Fig. 2

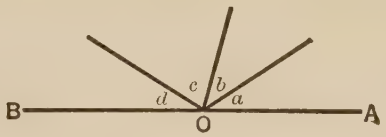
Fig. 1. 1. Complete the drawing to show the two supplements.
2. Follow the outline used for Fig. 1, § 93.

Fig. 2. $\measuredangle$ AOC and DOB are st. $\measuredangle$. Show $\angle s = \angle t$.

Put proofs in equation form, using the letters of the drawing.

Ex. AOB is a given angle. OC is perpendicular to OA, and OD is perpendicular to OB, OC and OD both lying on the *same* side of OB. Show (*a*) that $\angle COD = \angle AOB$ if $\angle AOB$ is acute, and (*b*) that $\angle COD$ is the supplement of $\angle AOB$ if $\angle AOB$ is obtuse.

95. *The sum of all the angles about a point on one side of a straight line through the point is a straight angle.*



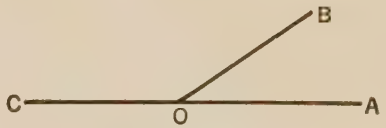
Prove: $\angle a + \angle b + \angle c + \angle d =$ a st. $\angle$.

1. What $\angle$ in the drawing is known to be a st. $\angle$?
2. Compare this $\angle$ with the required sum.

96. *The sum of all the angles about a point is equal to* 360°.

1. What is the relation between § 95 and § 96?
2. Make a suitable drawing and prove.

97. *If the sum of two adjacent angles equals a straight angle, their exterior sides form one straight line.*

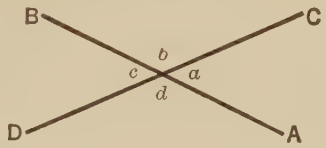


Given: Two adjacent angles AOB and BOC such that $\angle AOB + \angle BOC =$ a st. $\angle$.

1. What sides must be shown to form a st. line?
2. Name the st. $\angle$ given in the drawing.
3. What is the relation between the sides of such an $\angle$?
4. Hence make the conclusion.

Ex. ABC and $A'B'C'$ are two right triangles with hypotenuse $AB =$ hypotenuse $A'B'$ and leg $BC =$ leg $B'C'$. Show that if the two triangles are placed so that $B'C'$ coincides with BC and A and A' are on opposite sides of BC, a *single* isosceles triangle will be formed.

98. *When two straight lines intersect, the two opposite vertical angles formed are equal.*



1. Mention a pair of $\measuredangle$ to be proved equal.
2. Mention a pair of $\measuredangle$ involving this pair known to be equal.
3. Write step 2 in equation form and use Ax. 3.

Note the similarity in method here and in § 94, Fig. 2.

99. *The sum of two sides of a triangle is greater than the third side.*

Prove: $AC + CB > AB$.

See Postulate 2, p. 45.

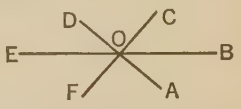
100. It should be noted that *the reliability of the conclusions just established in no instance depended on the accuracy of the drawing used or on the accuracy of the eye.* The conclusions follow directly from some definition, axiom, postulate, or combination of these.

The same general method used in establishing these conclusions will be used in the future in establishing other relations between the parts of various geometric figures. While reasonably accurate drawings should be made in all cases, nevertheless *in no instance should any conclusion be based on actual measurement of parts.*

Exercises — Group 22

1. Find an angle the sum of whose supplement and complement equals 124°.

2. If $\angle AOC$ = a rt. $\angle$ and $\angle EOF = 48°$, find all the other $\measuredangle$.

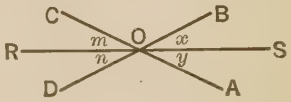


3. Two straight lines cross, forming two pairs of vertical angles. Show that if one of these angles is 90°, all are 90°.

4. In the drawing if $\angle x = \angle y$, show that $\angle m = \angle n$.

5. Show (*a*) algebraically and (*b*) by a drawing that if any acute or obtuse angle is subtracted from 360° and the remainder is diminished by the supplement of the given angle, the result is a straight angle.

6. If RS bisects $\angle AOB$, show that it bisects $\angle COD$; that is, if $\angle x = \angle y$, show that $\angle m = \angle n$.

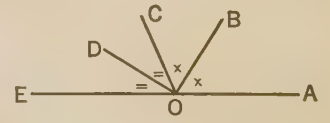


7. ABC is an isosceles triangle ($AC = BC$). AC is extended any distance to D and DB is drawn. Show that AD is longer than BD.

Sug. Use § 99 in $\triangle BCD$ and then use Ax. 8.

8. If BO bisects $\angle AOC$ and DO bisects $\angle COE$, show that $\angle BOD$ is a right angle.

9. A diameter is longer than any other chord.



Sug. Form a $\triangle$ containing the chord and two radii. Then use the suggestions in ex. 7.

PART TWO

DEMONSTRATIVE GEOMETRY

FORMAL ARRANGEMENT OF A PROOF

101. In the Introductory Course just finished, many simple geometric facts were discovered through observation and measurement. Some of these facts were established informally by a proof. There are many other facts of geometry whose truth cannot be so easily established. It is the purpose of Demonstrative Geometry to arrange in logical order the most important of these facts and to work out methods of discovery and proof.

102. A *theorem* is a statement that is to be proved.

For example: "When two straight lines intersect, the vertical angles are equal."

103. A *problem* is a construction to be made that satisfies certain given conditions.

For example: "To draw a perpendicular to a line at a point in the line."

104. Theorems and problems are often called *propositions*.

105. Proof. The process of reasoning whereby the truth of a theorem or the correctness of a construction is established is called a *proof* or a *demonstration*.

Every demonstration to be conclusive must justify each statement by a reason for it. The reason will be some statement that has been assumed or that has been proved previously. This suggests that steps in the demonstration should be kept distinct and arranged in logical order.

106. The theorems first considered will naturally be the ones that are most fundamental and that will be used most frequently. As the work advances, other theorems depending directly or indirectly on these will follow, and thus gradually there will be built up a system of propositions, each depending on some of those which precede, and in turn, perhaps, forming the foundation for some that follow.

107. Arrangement of a Demonstration. The formal demonstration of a theorem in geometry involves three steps:

(1) A statement (in terms of the figure used) of what is *given* — the *hypothesis.*

(2) A statement (in terms of the figure used) of what is *to be proved* — the *conclusion.*

(3) The *proof,* consisting of several steps, logically arranged, each based on the hypothesis or on some definition, axiom, postulate, or statement previously proved.

These three steps in the demonstration are usually indicated by the words — "**Given**," "**To prove**," "**Proof.**"

Thus, in § 97 the hypothesis is: "$\angle AOB + \angle BOC$ equals a straight angle," and the conclusion is: "line AOC is a straight line."

108. In arranging the proof it is best to number each step in order and to separate the statements from the reasons. Care should be taken to have corresponding steps in the "Statements" and "Reasons" columns numbered alike.

109. Pupils should carefully study the *analysis* in theorems where it is given, in order to learn how to discover the proof, before studying the formal proof. Desired results can be secured only by adhering strictly to this order.

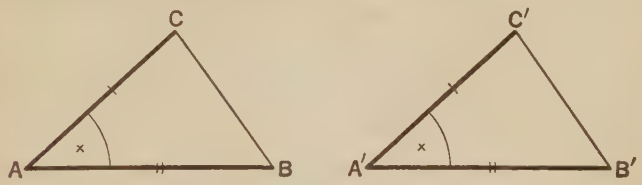
BOOK ONE

RECTILINEAR FIGURES

TRIANGLES

Proposition I. Theorem

110. *Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Given: $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $AC = A'C'$, and $\angle A = \angle A'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

The plan is to place one triangle upon the other and show that they can be made to coincide throughout.

Proof: Statements	Reasons
1. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that A falls on A' and side AB falls along side $A'B'$.	1. A geom. fig. may be freely moved in space without change in form or size.
2. Then AC will fall on $A'C'$.	2. $\angle A = \angle A'$ by hyp.
3. Also point B will fall on B'.	3. $AB = A'B'$ by hyp.
4. Also point C will fall on C'.	4. $AC = A'C'$ by hyp.
5. $\therefore$ CB must coincide with $C'B'$.	5. Only one st. line can be drawn between two points.
6. $\therefore$ $\triangle ABC \cong \triangle A'B'C'$.	6. Geom. figs. that can be made to coincide are congruent.

111. A *corollary* to a geometric statement is another geometric statement that is easily deduced from the given statement. In proving corollaries, however, drawings, complete statements, and reasons should generally be made as in the proof of a regular theorem.

112. COROLLARY. *Two right triangles are congruent if the two legs of one triangle are equal respectively to the two legs of the other.*

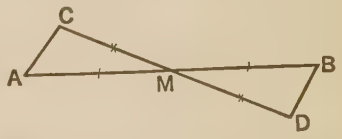
Ex. 1. What are the two points referred to in reason 5 of § 110?

Ex. 2. Name all equal parts in the triangles of § 110. (See § 70.)

Ex. 3. If $AM = MB$ and $CM = MD$, show $\triangle AMC \cong \triangle BMD$.

To what angle does $\angle C$ correspond?

To what line is AC equal? Why?



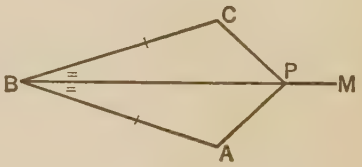
Ex. 4. Could step 2 of § 110 read "*so that* AC will fall on $A'C'$"? Explain.

Ex. 5. Given: BM the bisector of $\angle ABC$ and $BA = BC$.

Prove: $\triangle CBP \cong \triangle ABP$.

To what angle does $\angle CPB$ correspond? Why?

To what line is CP equal? Why?



Ex. 6. Are the hypotenuses equal in two right triangles if the legs are equal respectively? Why?

Ex. 7. Given: D the mid-point of AB and $CD \perp AB$.

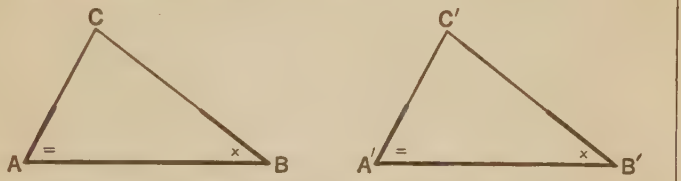
Prove: $\triangle ADC \cong \triangle BDC$.

To what line does AC correspond?

To what angle is $\angle ACD$ equal? Why?

Proposition II. Theorem

113. *Two triangles are congruent if a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*



Given: $\triangle ABC$ and $A'B'C'$ with $\angle A = \angle A'$, $AB = A'B'$, and $\angle B = \angle B'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

The plan is to place one triangle upon the other and show that they can be made to coincide throughout.

Proof: Statements	Reasons
1. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that AB falls on its equal $A'B'$, A on A', and B on B'.	1. A geom. fig. may be freely moved in space without change in form or size.
2. AC will extend along $A'C'$, C falling somewhere on $A'C'$.	2. $\angle A = \angle A'$ by hyp.
3. BC will extend along $B'C'$, C falling somewhere on $B'C'$.	3. $\angle B = \angle B'$ by hyp.
4. Then point C must fall on C'.	4. Two st. lines can intersect at but one point.
5. $\therefore \triangle ABC \cong \triangle A'B'C'$.	5. Geom. figs. that can be made to coincide are congruent.

114. Corollary. *Two right triangles are congruent if a leg and the adjacent acute angle of one are equal respectively to a leg and the adjacent acute angle of the other.*

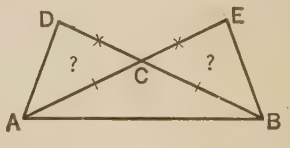
115. In referring to Theorem I in written work, to save time, the abbreviation s.a.s. may be used as a substitute for the theorem. Likewise the abbreviation a.s.a. may be used as a substitute for Theorem II. In oral work, however, these statements should be given in full.

Exercises — Group 23

1. What are the two lines referred to in reason 4 of § 113?

2. Name all the corresponding lines and angles that are equal in the triangles of § 113.

3. In the figure, $CA = CB$ and $CD = CE$. Show that $\triangle ADC \cong \triangle BEC$.



To what angle does $\angle D$ correspond?

To what line is AD equal? Why?

Show that $\triangle ADB \cong \triangle AEB$, and that $\angle ABD = \angle BAE$.

4. In the figure, $AO = BO$ and CA and DB are each $\perp SR$.

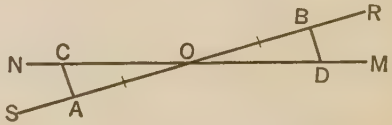
Prove: (a) $\triangle OAC \cong \triangle OBD$;

and hence

(b) $AC = DB$;

(c) $CO = OD$;

(d) $\angle C = \angle D$.

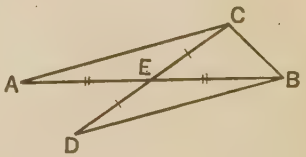


5. In the figure, E is the mid-point of AB and CE is extended its own length to D.

Show that $\triangle EDB \cong \triangle ECA$.

To what angle is $\angle ACD$ therefore equal?

To what angle is $\angle BAC$ equal?



6. In the figure, AD bisects $\angle BAC$ and $DC = DB$.

Prove: (a) $\triangle ABD \cong \triangle ACD$;

and hence

(b) $\angle B = \angle C$.

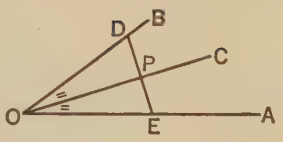
7. In the figure, CO bisects $\angle AOB$ and line $DE \perp CO$.

Show that $\triangle OPE \cong \triangle OPD$.

To what side does OD correspond? Why?

How do lines DP and PE compare in length?

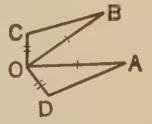
To what angle is $\angle BDP$ equal? Why?



8. In the figure, $\angle DOB$ and $\angle AOC$ are right angles; also $OD = OC$ and $OA = OB$.

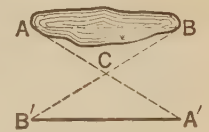
Show that $\triangle AOD \cong \triangle BOC$.

To what angle is $\angle A$ equal? Why?



9. If two lines are drawn bisecting each other and their end points are joined in order, prove that opposite pairs of triangles are congruent.

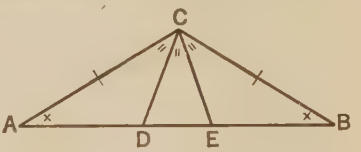
10. To find the distance between two points A and B *when both are accessible*, assume point C, make $CB' = CB$ and $CA' = CA$. Then $A'B' = AB$. Why?



11. In the triangle ABC, CD bisects $\angle C$ and $\perp AB$.

Prove: $\triangle ADC \cong \triangle BDC$.

What kind of triangle is $\triangle ABC$? Why?

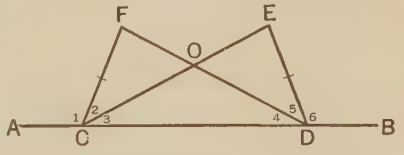


12. Given: $AC = BC$, $\angle A = \angle B$, and $\angle C$ trisected by CD and CE.

Prove: $CD = CE$.

13. Given: $\angle 1 = \angle 6$ and $CF = DE$.

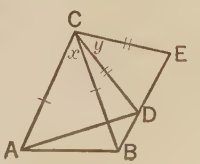
Prove: $\angle 3 = \angle 4$.



14. In the drawing for ex. 13, prove that $\triangle FOC \cong \triangle EOD$ and hence that $CO = DO$ and $FO = EO$.

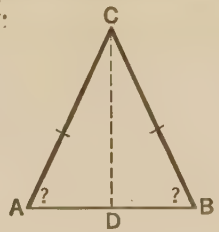
15. Given: $AC = BC$, $CD = CE$, and $\angle x = \angle y$.

Prove: $AD = BE$.



Proposition III. Theorem

116. *In any isosceles triangle the angles opposite the equal sides are equal.*



Given: Isosceles $\triangle ABC$ with $AC = BC$.
To prove: $\angle A = \angle B$.

The plan is to show that $\measuredangle A$ and B are corresponding angles of congruent triangles formed by bisecting $\angle C$.

Proof: Statements	Reasons
1. Draw CD to bisect $\angle C$.	1. Every $\angle$ has a bisector.
2. In $\triangle ADC$ and BDC, $AC = BC$.	2. By hyp.
3. $\angle ACD = \angle BCD$.	3. By cons.
4. $CD = CD$.	4. By identity.
5. $\therefore \triangle ADC \cong \triangle BDC$.	5. s.a.s.
6. $\therefore \angle A = \angle B$.	6. Corr. $\measuredangle$ of $\cong \triangle$.

117. Corollary. *An equilateral triangle is also equiangular.*

Discussion:

1. Why are $\angle A$ and $\angle B$ corr. $\measuredangle$?
2. Would congruent triangles be formed if CD were drawn:
   (*a*) From C to the mid-point of AB?
   (*b*) From $C \perp AB$? Explain.
3. Criticize the following methods of drawing CD:
   (*a*) From $C \perp AB$ at the mid-point of AB.
   (*b*) From C bisecting $\angle C$ and through the mid-point of AB.
   (*c*) From C bisecting $\angle C$ and $\perp AB$.

Auxiliary lines, such as CD, which are introduced to assist in establishing a proof, are usually represented by dash lines.

118. *In drawing any auxiliary line, care should be used not to put more conditions on the line than are necessary to determine the line.* Any additional condition must be proved.

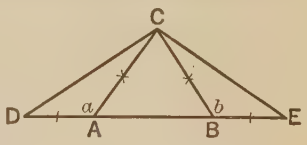
For example, in § 116, it would be incorrect to say "draw a line from C to the mid-point of AB and bisecting angle C," as you have seen in § 117, 3 (*b*). Either of the conditions determines a line. (See §§ 10 and 89.). Hence *one* of these conditions may be imposed on CD and then it must be *proved* that the other condition follows if we wish to use it.

Exercises — Group 24

1. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

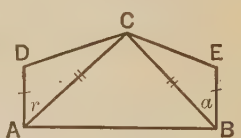
2. In the figure, $\triangle ABC$ is isosceles ($AC = BC$), and $AD = BE$.

Prove: (*a*) $\angle a = \angle b$; (*b*) $\triangle DCA \cong \triangle ECB$.



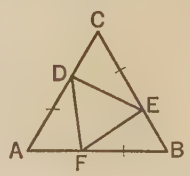
3. In the figure, $\triangle ABC$ is isosceles ($AC = BC$); also $AD = BE$ and each $\perp AB$.

Prove: (*a*) $\angle r = \angle a$; (*b*) $\triangle ADC \cong \triangle BEC$; and hence (*c*) $\angle D = \angle E$.



4. In the figure, $\triangle ABC$ is equilateral and $AD = CE = BF$. Show that $\triangle FAD \cong \triangle DCE \cong \triangle EBF$.

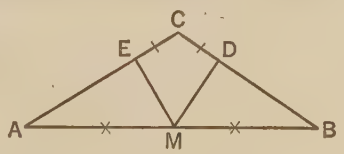
5. If two isosceles triangles are constructed on the same base, the line joining the two vertices forms two congruent triangles.



6. If the base of an isosceles triangle is trisected, prove that the lines joining the two points to the vertex are equal.

7. Given: $AC = BC$, $CE = CD$, and M the mid-point of AB.

Prove: $\triangle EAM \cong \triangle DBM$.



Proposition IV. Theorem

119. *Two triangles are congruent if the three sides of one are equal respectively to the three sides of the other.*

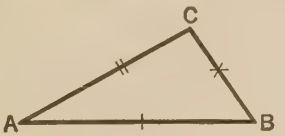


Fig. 1

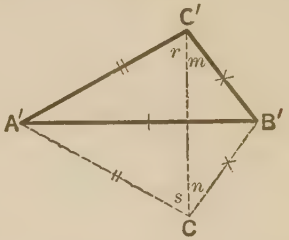


Fig. 2

Given: $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

The plan is to use s.a.s., with the aid of § 116.

Proof:

Statements	Reasons
1. Place $\triangle ABC$ so that one of the sides, AB, shall fall on its equal, $A'B'$, A upon A', B upon B', and so that C shall fall opposite C' (Fig. 2).	1. A geometric figure may be freely moved in space without any change in form or size.
2. Draw CC'.	2. Post 1, p. 45.
3. $\triangle A'CC'$ is isosceles.	3. $AC = A'C'$ by hyp.
4. $\therefore \angle s = \angle r$.	4. In an isosceles $\triangle$ the $\measuredangle$ opp. the $=$ sides are $=$.
5. Also $\triangle B'CC'$ is isosceles.	5. $BC = B'C'$ by hyp.
6. $\therefore \angle n = \angle m$.	6. Same as 4.
7. $\therefore \angle s + \angle n = \angle r + \angle m$.	7. If equals are added to equals, the sums are $=$.
8. $\therefore \angle A'CB' = \angle A'C'B'$.	8. The whole is equal to the sum of all its parts.
9. $\therefore \triangle A'B'C \cong \triangle A'B'C'$. That is, $\triangle ABC \cong \triangle A'B'C'$.	9. s.a.s.

Discussion:

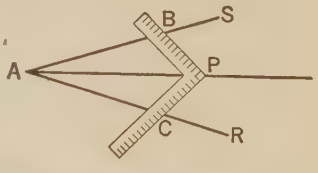
1. Place the ▲ so that the shortest pair of equal sides coincide. In what step does the proof now differ from the one above?
2. Suppose the two ▲ were rt. ▲.
3. What makes it inadvisable to try this proposition by superposition? Could that method be used? (See ex. 4, p. 124.)

120. In referring to Theorem IV in written work the abbreviation s.s.s. may be used for the theorem.

Exercises — Group 25

1. If three boards are nailed together in the form of a triangle, with one nail in each corner, would the frame formed be "rigid"?

2. To bisect an angle RAS with a carpenter's square, lay off $AB = AC$ and place the square so that $PB = PC$. Then AP bisects the angle. Why? Would the method be correct if $\angle BPC$ were not a right angle?



*** 3.** Show that the line from the vertex of an isosceles triangle to the mid-point of the base (*a*) bisects the vertex angle, (*b*) makes equal angles with (is perpendicular to) the base.

*** 4.** By means of § 119 prove that the method used in § 72 is correct. Show (1) $\triangle ACD \cong \triangle BCD$, then (2) $\triangle ACP \cong \triangle BCP$. Show also that the method used in §§ 73, 74, 75, and 76 is correct.

*** 5.** If two isosceles triangles have a common base, the line joining their vertices bisects the vertex angles and is the perpendicular bisector of the common base.

This line is called the *axis of symmetry* of the figure. Give other illustrations of an axis of symmetry.

6. By means of ex. 5 prove that the method used in §§ 72 and 76 is correct.

Proposition V. Theorem

121. *An exterior angle of a triangle is greater than either opposite interior angle.*

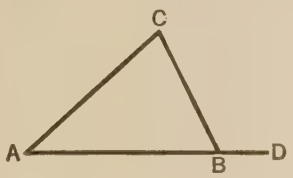


Fig. 1

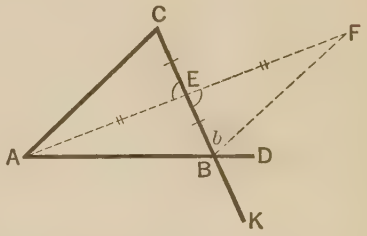


Fig. 2

Given: $\triangle ABC$ with exterior $\angle DBC$ and opposite interior $\measuredangle C$ and A.

To prove: $\angle DBC > \angle C$ or $\angle A$ (Fig. 1).

The plan is to show by congruent triangles that $\angle C$ and $\angle A$ each *equals a part of* $\angle DBC$.

Proof: Statements	Reasons
1. Let E be the mid-point of BC (Fig. 2).	1. Every line segment has a mid-point.
2. Draw AE and produce it to F, so that $EF = AE$. Draw FB.	2. Post. 1, 3.
3. Then $\triangle AEC \cong \triangle FEB$.	3. Why?
4. $\therefore \angle C = \angle b$.	4. Why?
5. But $\angle DBC > \angle b$.	5. The whole is greater than any of its parts.
6. $\therefore \angle DBC > \angle C$.	6. A quantity may be substituted for its equal in any process.

Similarly $\angle ABK\ (= \angle DBC) > \angle A$.

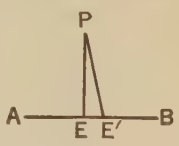
Proof in this latter case left to the pupil.

122. COROLLARY. *One and only one perpendicular can be drawn to a line from a point outside the line.*

(*a*) Ex. 6, p. 61, proved that one $\perp PE$ can be drawn from an external point P to a line AB.

(*b*) 1. Suppose PE' were a second $\perp$. Name all the angles that would be equal by § 91.

2. Show that § 121 is contradicted.



Discussion:

1. Why are $\measuredangle C$ and b corr. $\measuredangle$?
2. Must $\angle DBC$ be greater than $\angle ABC$? Must it be obtuse?
3. Make a statement to cover both cases ($\measuredangle A$ and C), showing what side of the given $\triangle$ is bisected in forming the two $\cong \measuredangle$.

123. Direct and Indirect Method of Proof. If two or more possibilities exist and we wish to prove that, under the particular conditions laid down in the hypothesis, one of these possibilities must be true, we may do this in either of two ways:

(1) We may single out this possibility and prove it directly with no reference to the other possibilities. This is called the *direct method of proof*.

(2) We may show that the other supposed possibilities lead to a contradiction of the hypothesis or of a previously proved theorem or of an axiom. This is called the *indirect method of proof* or *method by exclusion*. This method is sometimes referred to as the *reductio ad absurdum method*.

For example, there are three possible relations that may exist between two angles A and B: $\angle A$ may be less than $\angle B$, $\angle A$ may be greater than $\angle B$, $\angle A$ may be equal to $\angle B$. In Th. III, § 116, under the conditions in the hypothesis, we proved directly, with no reference to the other possibilities, that $\angle A = \angle B$; but we could show instead that the other two possibilities, namely, that $\angle A < \angle B$ or $> \angle B$, would lead to contradictions of some sort, in which case the only possibility left would be that $\angle A = \angle B$, which accordingly must be true.

124. No general rule can be given as to when the indirect method should be used. It may be tried when other methods fail. It is frequently applied to prove converse propositions.

125. A *converse proposition* is a proposition formed from another proposition by interchanging its hypothesis and conclusion.

For example, the converse of Th. III would be: "If two angles of a triangle are equal, the sides opposite those angles are equal."

Every theorem therefore has a converse, but few of them are true. Th. VI, which is the converse of Th. III, is true.

Exercises — Group 26

1. Show how the indirect method was used in §§ 88 and 89.

(*a*) In each case what were "the two possibilities"?

(*b*) Which possibility had to be eliminated?

(*c*) In each case what axiom was violated by the second possibility? What conclusion followed? (Compare § 123.)

2. The converse of Th. I, p. 53, would be: "If two triangles are congruent, then any two sides and the included angle of one are equal respectively to two sides and the included angle of the other." Is this converse true? (See § 65.)

3. State the converses of Th. II and Th. IV, pp. 55, 60. Are they true?

4. State the converse of each of the following and determine whether it is true:

(*a*) An acute angle is an angle less than a right angle.

(*b*) An obtuse angle is less than a straight angle.

(*c*) If two adjacent angles have their exterior sides lying in one straight line, the two adjacent angles are supplementary.

(*d*) If the sum of two acute angles is a right angle, the two acute angles are complementary.

(*e*) If two angles are right angles, the two angles are equal.

(*f*) If a triangle is isosceles, the bisector of the vertex angle of the triangle is perpendicular to the base.

(*g*) If two triangles are congruent, the three angles of one are equal respectively to the three angles of the other.

Proposition VI. Theorem

126. *If two angles of a triangle are equal, the sides opposite those angles are equal.* (Converse of Th. III.)

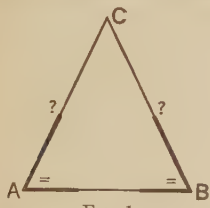


Fig. 1

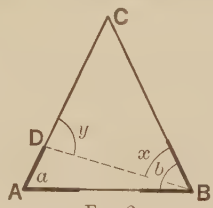


Fig. 2

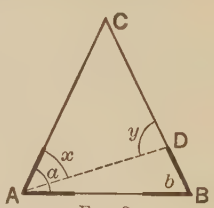


Fig. 3

Given: $\triangle ABC$ in which $\angle B = \angle A$.

To prove: $CA = CB$.

The plan is to use the indirect method.

Analysis:

1. What are the three possible relations between CA and CB?
2. Which do we wish to prove true according to the theorem?
3. Hence which relations must we prove to be impossible?
4. If $CA > CB$, we can have Fig. 2 in which $CD = CB$.
5. Compare $\angle b$ with $\angle x$ and $\angle y$; compare $\angle y$ with $\angle a$.
6. Hence compare $\angle b$ with $\angle a$. (See Ax. 9.)
7. Is this relation possible here? Then is step 4 possible?
8. If $CA < CB$, we can have Fig. 3 in which $CD = CA$.
9. Show that this drawing is impossible.
10. Then what must be the relation between CA and CB?

Proof:

Left to the pupil.

127. Corollary. *An equiangular triangle is also equilateral.*

Discussion:

1. Could we obtain $\cong$ $\triangle$ as in Th. III, p. 58, by drawing some line from C to AB? (See § 118.) Discuss all possibilities.
2. State the converse of § 127. Is this converse true?

128. Propositions I–VI suggest additional methods for proving a pair of lines or a pair of angles equal.

I. *To prove a pair of lines equal:*

(*a*) Show that they are corresponding lines in two congruent triangles.

(*b*) Show that they are sides opposite equal angles in the *same* triangle.

II. *To prove a pair of angles equal:*

(*a*) Show that they are corresponding angles in two congruent triangles.

(*b*) Show that they are angles opposite equal sides in the *same* triangle.

129. In working out exercises that will be given from time to time, it is very desirable that the pupil acquire a definite method of attack so that when an exercise is assigned, he may have some system to his procedure. There is no general method that can be used to solve exercises in geometry. Some suggestions, however, may be given that will tell the pupil " how to start " and thus give system and direction to his efforts.

1. *Draw carefully a figure representing what is given.* (The figure should be as general as possible.)

2. *Determine specifically what is to be proved.* (Usually it reduces to proving a pair of lines or a pair of angles equal.)

3. *Recall the various ways you have had for proving this.*

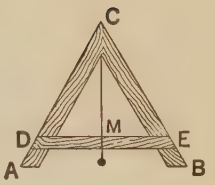
4. *Determine which of these ways is the best for the case at hand.* (This may be done by elimination if it is not evident at once.)

5. *Determine what parts you have and what parts, if any, you lack to use this method.*

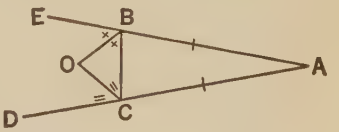
6. *Determine how you may get the parts that are lacking.*

Exercises — Group 27

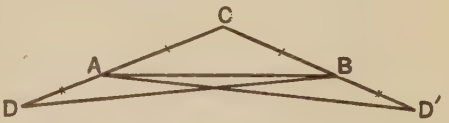
1. The A-level used by carpenters consists of three pieces of board nailed together in the form of a capital letter A. A plumb bob is hung from a hook at C. Show that any surface upon which A and B rest is level ($\perp$ to the plumb line) when this line lies over M, the mid-point of DE.



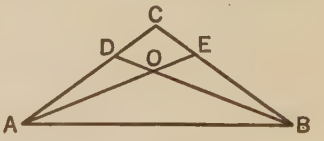
2. Given: $AB = AC$ and OB and OC the bisectors of $\angle EBC$ and $\angle DCB$ respectively.
Prove: $OB = OC$.



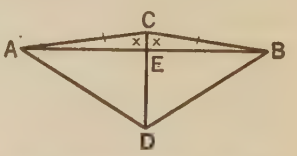
3. Given: $AC = BC$ and $AD = BD'$.
Prove: $\angle D = \angle D'$.



4. Given: $AC = BC$ and $DC = EC$.
Prove: $\triangle OAB$ is isosceles.



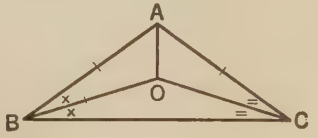
5. Given: $AC = BC$ and DC bisector of $\angle ACB$.
Prove: (*a*) $AE = BE$;
(*b*) $\triangle ABD$ is isosceles.



***6.** The perpendicular bisector of the base of an isosceles triangle passes through the vertex of the triangle.

Sug. Use the indirect method. See ex. 3, p. 61; also § 92.

7. Given: $AB = AC$, and OB and OC bisecting $\angle CBA$ and $\angle BCA$ respectively.
Prove: (*a*) $\triangle OAB \cong \triangle OAC$;
(*b*) OA bisects $\angle BAC$.

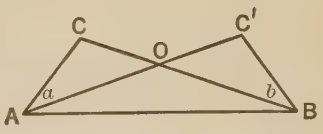


8. Given: $AC = BC'$ and $AC' = BC$.

Prove: (*a*) $\angle C = \angle C'$;

(*b*) $\angle a = \angle b$;

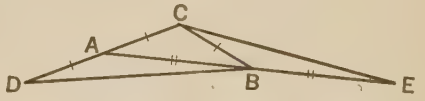
(*c*) $\triangle AOC \cong \triangle BOC'$.



9. In ex. 8 join C and C' and prove that $\triangle OCC'$ is isosceles.

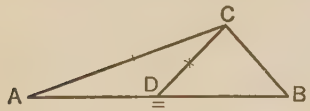
10. Given: $AB = BE$ and $CB = CA = AD$.

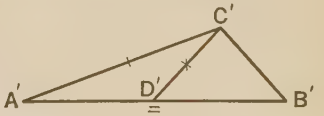
Prove: $\triangle ABD \cong \triangle EBC$.



11. Given: $AC = A'C'$, $AB = A'B'$, $CD = C'D'$, D and D' being the mid-points of AB and $A'B'$ respectively.

Prove: $\triangle ABC \cong \triangle A'B'C'$. (Use §110 after using § 119.)

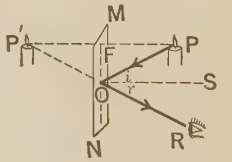




12. Prove that two triangles are congruent if a side, an adjacent angle, and the bisector of that angle in one triangle are equal respectively to the corresponding parts in the other.

***13.** It is shown in physics that when a ray of light PO strikes a plane mirror MN at O, it is reflected along OR in such a way that PO and OR make equal angles with the $\perp OS$; that is, $\angle i = \angle r$. The light from P would therefore strike the eye at R.

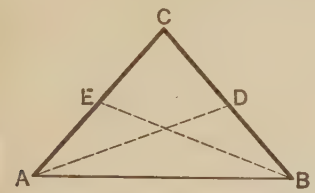
If RO extended meets the $\perp PF$ produced at P', show that $P'F = PF$.



This explains why an object, reflected in a plane mirror, appears to the eye to be located the same distance behind the mirror as it actually is in front of the mirror.

130. Alternative Proofs for Theorems III and VI. In the proofs on p. 69, note the similarity of procedure in each case.

Theorem III



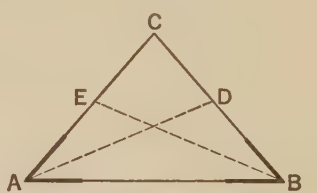
Given: $\triangle ABC$, with $AC = BC$.

To prove: $\angle BAC = \angle ABC$.

Analysis:

1. Draw lines from A and B, bisecting the *equal sides*.
2. What $\triangle$ must be proved $\cong$ to get *angle* A = *angle* B?
3. What *sides* are = by cons.?
4. What *side* and *angle* are lacking to make them $\cong$?
5. Can these *sides* be proved = by another pair of $\triangle$?
6. Do these $\triangle$ also help to get the *angles* equal?
7. Hence are the $\triangle$ in $2 \cong$?
8. Does *angle* A = *angle* B?

Theorem VI



Given: $\triangle ABC$, with $\angle BAC = \angle ABC$.

To prove: $AC = BC$.

Analysis:

1. Draw lines from A and B bisecting the *equal angles*.
2. What $\triangle$ must be proved $\cong$ to get *side* AC = *side* BC?
3. What *angles* are = by cons.?
4. What *angle* and *side* are lacking to make them $\cong$?
5. Can these *angles* be proved = by another pair of $\triangle$?
6. Do these $\triangle$ also help to get the *sides* equal?
7. Hence are the $\triangle$ in $2 \cong$?
8. Does *side* AC = *side* BC?

Proof: Let the pupil write the complete proof in each case.

Discussion:

1. In Th. III could E and D be taken equal distances from A and B (a) on AC and BC? (b) on AC and BC extended through C? (c) on CA and CB extended through A and B respectively? Make drawings in each case and prove.
2. Make a generalization of possible constructions for Th. III.
3. Make a corresponding generalization for Th. VI and verify.
4. In step 4 in each case there are two possible combinations. Why is one at once eliminated?

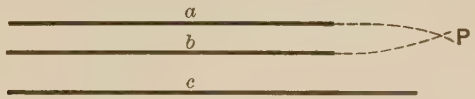
PARALLEL LINES

131. *Two lines are parallel* if they lie in the same plane and do not meet however far they are produced.

132. PARALLEL POSTULATE. *Through a given point one and only one straight line can be drawn parallel to a given line.*

This postulate may be stated as follows: *Two intersecting straight lines cannot both be parallel to the same straight line.*

133. COROLLARY. *If two straight lines are parallel to a third straight line, they are parallel to each other.*



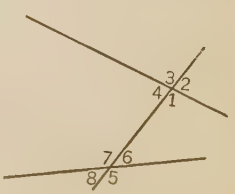
Given: Lines a and b each $\parallel c$.
To prove: $a \parallel b$.

Analysis:

1. a and b must either meet or be $\parallel$. Which do we wish to prove true?
2. Show that the other assumption contradicts § 132.
3. Hence, what conclusion follows?

134. A *transversal* is a line that intersects two or more other lines.

135. When two lines are cut by a transversal, eight angles are formed. These angles are classified as follows:



$\measuredangle$ 1, 4, 6, 7 are *interior angles;*
$\measuredangle$ 2, 3, 5, 8 are *exterior angles;*
$\measuredangle$ 4 and 6 or $\measuredangle$ 1 and 7 are *alternate interior angles;*

$\measuredangle$ 2 and 8 or $\measuredangle$ 3 and 5 are *alternate exterior angles;*

$\measuredangle$ 1 and 5, $\measuredangle$ 2 and 6, $\measuredangle$ 4 and 8, $\measuredangle$ 3 and 7 are *corresponding angles* (also called *exterior-interior angles*).

136. The study of parallel lines will concern itself chiefly with two things:

1. A study of the relations that must exist between these angles *if the two lines* ***are to be made*** *parallel;* and conversely:

2. A study of the relations that must exist between these angles *if the two lines* ***are*** *parallel.*

Ex. 1. If a straight line cuts one of a series of parallel lines, it must, if extended, cut all the parallel lines.

Ex. 2. Show that two parallel lines and a transversal of these lines could never form a triangle.

Ex. 3. Show by § 122, using the indirect method, that two lines perpendicular to the same line cannot meet; that is, that they are parallel.

Ex. 4. Show, by ex. 3, that no triangle can have in it two right angles.

Ex. 5. Show that if a transversal is drawn to two lines which intersect, the alternate angles formed cannot be equal.

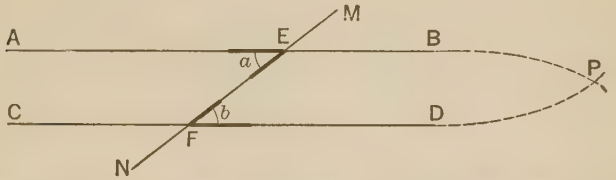
Ex. 6. If a transversal is drawn to two intersecting lines, will the corresponding angles formed be equal? Explain.

Ex. 7. In the figure of § 135, assuming that the two lines are not parallel, can:

(*a*) $\angle 4 = 60°$ and $\angle 6 = 50°$? Explain.
(*b*) $\angle 4 = 50°$ and $\angle 6 = 70°$? Explain.
(*c*) $\angle 4 = 80°$ and $\angle 6 = 80°$? Explain.
(*d*) $\angle 2 = 75°$ and $\angle 8 = 50°$? Explain.
(*e*) $\angle 3 = 100°$ and $\angle 7 = 100$? Explain.
(*f*) $\angle 2 = \angle 8$? Explain.

Proposition VII. Theorem

137. *When two straight lines are cut by a transversal, if a pair of alternate interior angles are equal, the two straight lines are parallel.*



Given: Two st. lines AB and CD cut by the transversal MN at points E and F respectively so that $\angle a = \angle b$.

To prove: $AB \parallel CD$.

Analysis:

1. If AB and CD did meet, what kind of figure would they form with EF?
2. Then what relation would exist between $\angle a$ and $\angle b$, by § 121?
3. Is this possible? Why? Then what is your conclusion?

Proof:

Left to the pupil.

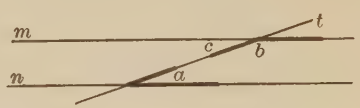
138. Corollary 1. *When two straight lines are cut by a transversal, if two corresponding angles are equal, the two straight lines are parallel.*

Show from the pair of angles given equal that two alternate interior angles are equal, and apply § 137.

139. Corollary 2. *Two lines perpendicular to the same straight line are parallel.*

140. Corollary 3. *When two straight lines are cut by a transversal, if the sum of the two interior angles on the same side of the transversal is equal to a straight angle, the two straight lines are parallel.*

Given: $\angle a + \angle b$ = a st. $\angle$.
To prove: $m \parallel n$.



Analysis:

1. To prove $m \parallel n$, what $\measuredangle$ must be shown equal?
2. What relation has each of these $\measuredangle$ to $\angle b$? Why?
3. Hence, are the $\measuredangle$ equal? Why? Are the lines $\parallel$? Why?

Discussion:

In § 137 suppose that $\angle FEB$ were given equal to $\angle CFE$. Might not the lines meet on the other side of MN?

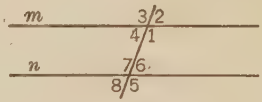
Ex. 1. Show that in the drawing of § 135:

(*a*) If any pair of odd-numbered angles or any pair of even-numbered angles (not including vertical angles) are equal, the two lines are parallel.

(*b*) If any pair of odd- and even-numbered angles (not including adjacent angles) are supplementary, the two lines are parallel.

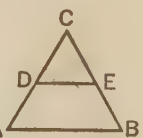
Ex. 2. Is $m \parallel n$ if:

(*a*) $\angle 4 = 70°$, $\angle 6 = 70°$?
(*b*) $\angle 3 = 110°$, $\angle 7 = 110°$?
(*c*) $\angle 2 = 70°$, $\angle 8 = 70°$?
(*d*) $\angle 1 = 110°$, $\angle 8 = 60°$?
(*e*) $\angle 1 = 110°$, $\angle 6 = 70°$?



Ex. 3. In § 121 show that $BF \parallel AC$.

Ex. 4. If $\triangle ABC$ and DEC are both equilateral, show that $DE \parallel AB$.



Ex. 5. Consider the case where E and D are on the prolongation of AC and BC through C.

Proposition VIII. Theorem

141. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.* (Converse of Th. VII.)

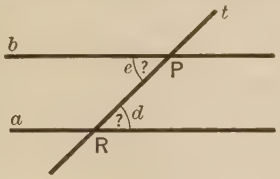


Fig. 1

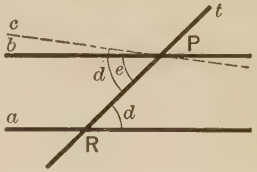


Fig. 2

Given: Two ∥ lines a and b cut by the transversal t at points R and P, forming the alternate interior ∡s d and e.

To prove: $\angle d = \angle e$.

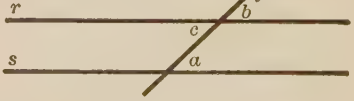
Analysis:

1. Either $\angle d = \angle e$ or $\angle d$ does not $= \angle e$. Which do we wish to prove true? Hence, which must be shown impossible?
2. If $\angle d$ did not $= \angle e$, it would be possible to draw a line c through P making an $\angle$ at $P = \angle d$ (Fig. 2).
3. Then what relation would c have to a? Why?
4. Is this relation possible? (See § 132.)
5. Then is step 2 possible? What is your conclusion?

Proof:

Left to the pupil.

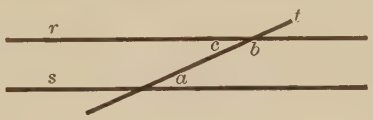
142. Corollary 1. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*



Analysis:

1. What two angles are to be proved equal?
2. To what angle is each one equal? Why?
3. Then why are the two angles in question equal?

143. COROLLARY 2. *If two parallel lines are cut by a transversal, the sum of the two interior angles on the same side of the transversal is equal to a straight angle.*



Analysis:

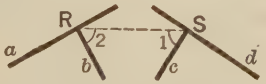
1. What two angles are known to form a st. $\angle$? Why?
2. What two angles do we wish to prove equal to a st. $\angle$?
3. What condition would make these two pairs of $\measuredangle$ equal?
4. Is this condition true in the drawing? Why?

144. COROLLARY 3. *When two straight lines are cut by a transversal, if the sum of the interior angles on the same side of the transversal does not equal a straight angle, the two lines are not parallel.*

Use § 143 and the indirect method.

145. COROLLARY 4. *A straight line perpendicular to one of two parallel lines is perpendicular to the other also.*

146. COROLLARY 5. *Lines perpendicular to non-parallel lines are not parallel.*



Given: $b \perp a$, $c \perp d$, and a not $\parallel d$.
To prove: b is not $\parallel c$.

Analysis:

1. What condition mentioned in § 144 proves two lines not $\parallel$?
2. If RS is the transversal, is this condition satisfied? Explain.

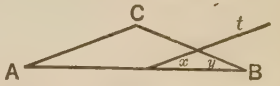
Discussion:

1. In Th. VIII, and corollaries, where more than one combination of angles might be given, you should work out each combination.
2. In the theorem, how would you prove the other pair of alternate interior angles equal?
3. In § 146, where is the fact used that a is not $\parallel d$?

Exercises — Group 28

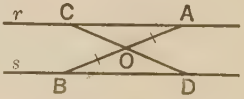
1. State the converses of Cor. 1–3, Th. VII. Are they true?

2. Given: $AC = BC$ and $t \parallel AC$.
Prove: $\angle x = \angle y$.



3. If a line is drawn parallel to the base of an isosceles triangle, cutting the equal sides, it forms with those sides another isosceles triangle.

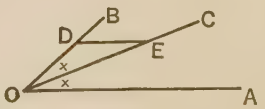
4. Given: $r \parallel s$, O the mid-point of AB, and CD any other line through O.
Prove: $CO = DO$.



5. In ex. 4, if $AO = BO$ and $CO = DO$, prove $r \parallel s$ and $CA = BD$.

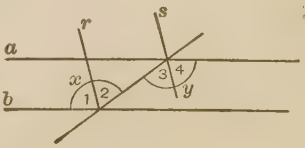
6. State the hypothesis and the conclusion of the converse of ex. 5. Prove this converse.

7. Given: OC bisector of $\angle AOB$ and $DE \parallel OA$.
Prove: $\triangle OED$ is isosceles.



8. In the figure, AB intersects DC and AC and BD are $\perp DC$.
Prove: $\angle A = \angle B$.

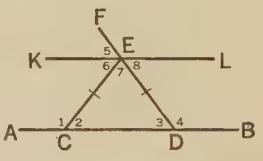
9. Given: $a \parallel b$, r bisecting $\angle x$, and s bisecting $\angle y$.
Prove: $r \parallel s$.



10. If a line is drawn through the vertex of an isosceles triangle parallel to the base, it bisects the exterior angle at the vertex.

11. If the bisector of the exterior angle opposite the base of a triangle is parallel to the base, the triangle is isosceles. (Converse of ex. 10.)

12. If $EC = ED$, $KL \parallel AB$, and $\angle 5 = 52°$, find all the other numbered angles of the drawing.



13. An exterior angle at any vertex of a triangle is equal to the sum of the two opposite interior angles.

SUG. Through the vertex draw a line $\parallel$ the opposite side.

14. Given: Quad. $ABCD$, $AB \parallel DC$ and $AD \parallel BC$.

Prove: $AB = DC$ and $AD = BC$.

SUG. Draw DB.

15. Given: $DC \parallel AB$, $AD \parallel BC$, and $DE = BF$.

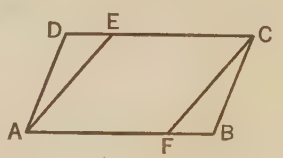
Prove: $AE \parallel FC$.

16. In a polygon of four sides, if three of the angles are right angles the fourth angle also is a right angle.

17. Given: $AB \parallel DC$, $AD \parallel BC$, and $AE \parallel FC$.

Prove: $\triangle FBC \cong \triangle EDA$.

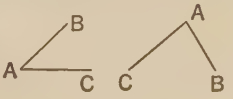
SUG. Draw EF.



18. In ex. 17, if $FB = DE$, $AB \parallel DC$, and $AD \parallel BC$, prove that $\triangle FBC \cong \triangle EDA$.

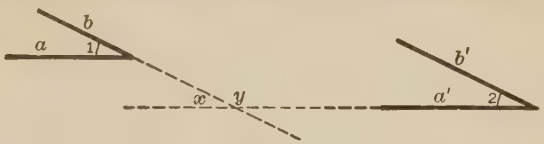
147. Right Side and Left Side of an Angle. Viewed from the vertex, the side of an angle at the right is called the *right side* of the angle. The side at the left is called the *left side* of the angle.

Thus, in the drawings, AC is the right side of $\angle A$, and AB is the left side.



Proposition IX. Theorem

148. *If two angles have their sides parallel, right side to right side and left side to left side, the angles are equal.*



Given: $\angle 1$ and $\angle 2$ with their sides a, b and a', b' parallel, $a \parallel a'$ and $b \parallel b'$.

To prove: $\angle 1 = \angle 2$.

Analysis:

1. Which of these lines if extended would meet? Why?
2. Then find an angle *known* to $= \angle 1$. Where? Why equal?
3. How does this angle compare with $\angle 2$? Why?
4. Hence how do $\angle 1$ and $\angle 2$ compare? Why?

Proof:

Left to the pupil.

149. Corollary. *If two angles have their sides parallel, right side to left side and left side to right side, the angles are supplementary.*

Analysis:

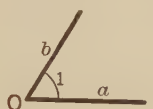
1. Which of these lines if extended would meet? Why?
2. Find an angle *known* to be sup. to $\angle 1$. Where? Why sup.?
3. How does this angle compare with $\angle 2$? Why? (See § 148.)
4. Hence how do $\angle 1$ and $\angle 2$ compare? Why?

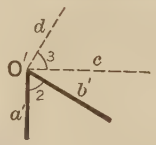
Discussion:

Pupils should discover various combinations of angles that might be used in the above proofs.

Proposition X. Theorem

150. *If two angles have their sides perpendicular, right side to right side and left side to left side, the angles are equal.*





Given: $\angle 1$ and $\angle 2$ with their sides a, b and a', b' perpendicular, $a \perp a'$ and $b \perp b'$

To prove: $\angle 1 = \angle 2$.

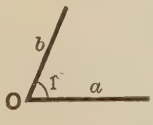
Analysis:

1. Form $\angle 3$ at O' *known* to $= \angle 1$, by § 148.
2. Point out the rt. $\measuredangle$ at O'. Give reasons for each.
3. How does $\angle 3$ compare with $\angle 2$? (See § 93.)
4. Hence how do $\angle 1$ and $\angle 2$ compare? Why?

Proof:

Left to the pupil.

151. Corollary. *If two angles have their sides perpendicular, right side to left side and left side to right side, the angles are supplementary.*



Analysis:

1. Construct an $\angle$ at O' *known* to be sup. to $\angle 2$.
2. How does this $\angle$ compare with $\angle 1$? Why? (See § 150.)
3. Hence how do $\angle 1$ and $\angle 2$ compare? Why?

Discussion:

Pupils should discover various combinations of angles that might be used in the above proofs.

Proposition XI. Theorem

152. *The sum of the angles of a triangle is a straight angle.*

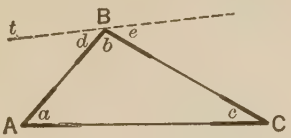


Fig. 1

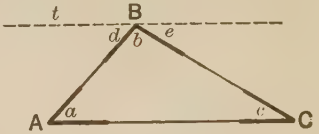


Fig. 2

Given: $\triangle ABC$ with its three angles a, b, c.

To prove: $\angle a + \angle b + \angle c =$ one st. $\angle$.

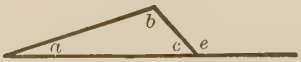
Analysis:

1. Any line through B, such as t, would give an $\angle$ sum equal to one st. $\angle$. For example, $\angle d + \angle b + \angle e =$ one st. $\angle$ (Fig. 1).
2. What $\angle$ sum do we wish to prove equal to one st. $\angle$?
3. What relation between the $\measuredangle$ of the drawing would make these two sums identical?
4. Does this suggest how to draw line t?

Proof:

Statements	Reasons
1. Through B draw a line $t \parallel AC$ (Fig. 2).	1. § 132 *
2. Then $\angle d + \angle b + \angle e =$ one st. $\angle$.	2. § 95
3. But $\angle d = \angle a$, and $\angle e = \angle c$.	3. § 141
4. $\therefore \angle a + \angle b + \angle c =$ one st. $\angle$, by substituting $\angle a$ for $\angle d$ and $\angle c$ for $\angle e$.	4. Ax. 8

153. Corollary 1. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*



1. $\angle a + \angle b + \angle c = \angle c + \angle e$. Ax. 1.
2. $\therefore \angle a + \angle b = \angle e$. Ax. 3.

* In the demonstration of a proposition pupils should not refer to axioms, postulates, previous theorems, etc. by number, but should state them in full.

154. COROLLARY 2. *In any triangle there can be but one right angle or one obtuse angle.*

155. COROLLARY 3. *In any right triangle the two acute angles are complementary.*

156. COROLLARY 4. *If an acute angle of one right triangle equals an acute angle of another right triangle, the remaining acute angles of the two triangles are equal.*

157. COROLLARY 5. *Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.*

Show that the two triangles have a.s.a.

158. COROLLARY 6. *Two right triangles are congruent if a leg and either acute angle of one are equal respectively to a leg and the corresponding acute angle of the other.*

159. COROLLARY 7. *If two angles of one triangle are equal respectively to two angles of another, then the third angle of the first is equal to the third angle of the second.*

160. COROLLARY 8. *Two triangles are congruent if a side and any two angles of one are equal respectively to a side and two angles of the other.*

Show that in all such cases the triangles have two angles and the included side of one equal to two angles and the included side of the other. The abbreviation s.a.a. may be used for Cor. 8.

161. COROLLARY 9. *Each angle of an equilateral triangle is* 60°.

Discussion:

Could t have been drawn through any vertex $\parallel$ the opposite side?

Exercises — Group 29

1. An exterior angle of a triangle is 124°. An opposite interior angle is 72°. Find all the angles of the triangle.

2. Two angles of a triangle are 70° and 50°. Write two sets of exterior angles of this triangle.

3. The vertex angle of an isosceles triangle is 40°. Find the base angles.

4. An exterior angle at the base of an isosceles triangle is 130°. Find the vertex angle.

5. The angles of a triangle are in the ratio 3 : 4 : 5. What are the angles?

6. Two angles of a triangle are 84° and 63°. What is the third angle?

* **7.** In a right triangle if one acute angle is double the other, what is the size of each? Explain.

This triangle is frequently referred to as the *60°–30° triangle.*

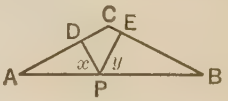
Exercises — Group 30

1. Prove Th. XI, p. 80, by using the following drawings:

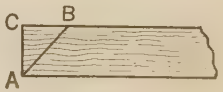
* **2.** Show that the sum of the four angles of any quadrilateral is equal to two straight angles.

3. Given: $AC = BC$, $PD \perp AC$, $PE \perp BC$.

Prove: $\angle x = \angle y$.

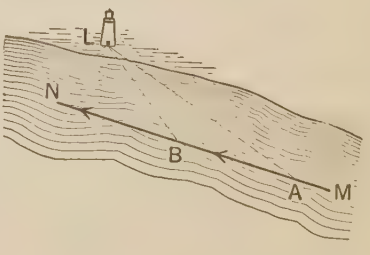


4. Show how to saw off a board at an angle of 45°.

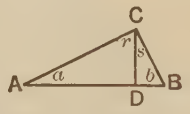


* **5.** Prove that if the two vertex angles of two isosceles triangles are equal, the other angles must be equal.

6. MN is the direction in which a ship is sailing and L is a lighthouse. At the position A the $\angle BAL$ is measured. The ship sails on and position B is noted where $\angle NBL = 2\angle A$. The distance AB being known from the ship's log, show that the distance BL is known.

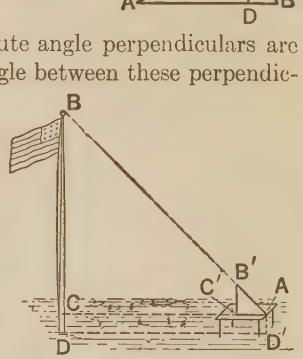


This is known as "doubling the angle on the bow" and is used in finding the distance from a ship to any point on land.



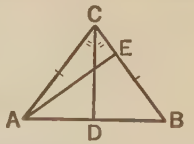
* **7.** Given: Rt. $\triangle ABC$, with $CD \perp AB$.
Prove: $\angle r = \angle b$, $\angle s = \angle a$.

8. If from any point outside an acute angle perpendiculars are drawn to the sides of the angle, the angle between these perpendiculars is equal to the given angle.



9. To find the height of a flagpole DB, a large 45° right triangle is placed on a horizontal support located at a point where the top, B, of the pole is just visible looking along AB', the hypotenuse of the triangle. Show that $BC = CA$ and that hence the length of the pole BD can be found.

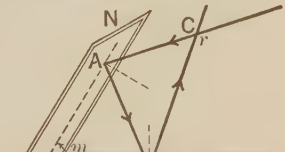
10. The angle between the bisectors of the base angles of an isosceles triangle is equal to the exterior angle formed by extending the base.



11. Given: $AC = BC$, CD bisecting $\angle C$, and $AE \perp BC$.
Prove: $\triangle ADC$ and ABE have their $\measuredangle$ equal respectively.

* **12.** Prove that two isosceles triangles are congruent if any side and any angle in one are equal to the corresponding side and angle in the other. Consider the various combinations.

13. Two plane mirrors MO and NO form an acute angle m with each other. A ray of light CA is reflected by NO so as to strike MO, and is again reflected by MO, crossing its first path at C. Show that $\angle r = 2\angle m$. (See ex. 13, p. 68.)

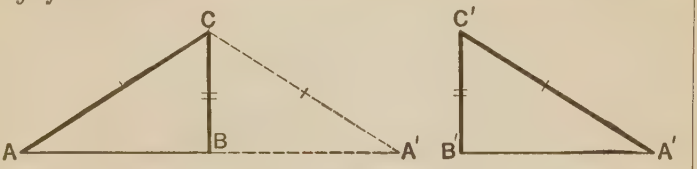


This is the principle involved in the use of the *sextant*.

14. If the path of a ray of light, reflected as in ex. 13, forms an equilateral triangle, what is the angle m between the mirrors?

Proposition XII. Theorem

162. *Two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.*



Given: Rt. $\triangle ABC$ and $A'B'C'$ with $AC = A'C'$ and $BC = B'C'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

Analysis:

1. Two possible ways are suggested: (*a*) by superposition, (*b*) by a previous theorem. Which seems more probable? Why?
2. What two possible ways of placing the $\triangle$ are suggested by the hypothesis?
3. If placed as in the drawing, describe the position of AB and BA'. Explain.
4. Then what kind of figure is formed? Why?

Let the pupil complete the analysis and give the proof.

163. Corollary. *The perpendicular from the vertex to the base of an isosceles triangle bisects the base and also the vertex angle.*

Discussion:

1. Why is it necessary to prove that ABA' is a straight line?
2. Place the $\triangle$ so that the hypotenuses coincide and the right angles fall on opposite sides and prove the theorem, using §§ 116, 126, 112 in order.
3. Can the proposition be proved by superposition? (See ex. 3, p. 124.)

UNEQUAL LINES AND ANGLES

Proposition XIII. Theorem

164. *If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side.*

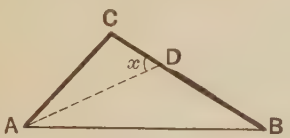


Fig. 1

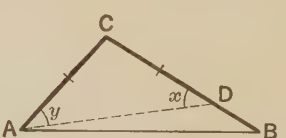


Fig. 2

Given: $\triangle ABC$ in which $BC > AC$.
To prove: $\angle A > \angle B$.

Analysis:

1. A line from A to any point D in AB would form an $\angle x > \angle B$ (Fig. 1). Explain.
2. If $\angle x = \angle BAC$ or $=$ a part of $\angle BAC$, the theorem is proved. Explain.
3. Can D be selected so that either of these conditions is true?

Proof:

Statements	Reasons
1. On CB lay off $CD = CA$ (Fig. 2).	1. Post. 3
2. Draw AD.	2. Post. 1
3. Show that $\angle BAC > \angle y$; then $\angle BAC > \angle x > \angle B$.	

Let the pupil complete the proof as outlined in step 3.

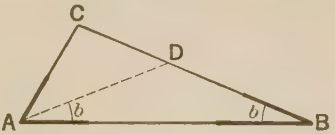
Discussion:

1. Explain why the construction in step 1 is possible.
2. Is $\angle A$ necessarily greater than $\angle C$ also? Explain.
3. Suppose CB were the hypotenuse of a rt. $\triangle$.

165. When two magnitudes are given unequal, *marking off the less upon the greater often leads to a possible line of proof.*

Proposition XIV. Theorem

166. *If one angle of a triangle is greater than a second angle, the side opposite the first angle is greater than the side opposite the second angle.* (Converse of Th. XIII.)



Given: $\triangle ABC$ in which $\angle A > \angle B$.

To prove: $BC > AC$.

Analysis:

1. See § 165. How is AD drawn?
2. Show that $BC = BD + DC = (AD + DC) > AC$.

Proof:

Left to the pupil.

167. Corollary. (*a*) *The perpendicular is the shortest line that can be drawn from a given point to a given line.*

Show that it is shorter than any other line from the point to the given line.

(*b*) Conversely, *if a line is the shortest line that can be drawn from a given point to a given line, it is the perpendicular from the point to the given line.*

Use the indirect method. (See § 122.)

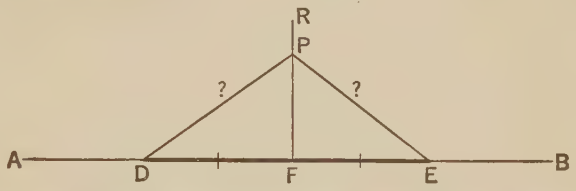
Ex. Prove Th. XIV by the indirect method:

1. What are the three possible relations between BC and AC?
2. Which do we wish to prove true?
3. Show that each of the other possibilities leads to a contradiction of the hypothesis.

This proof may be substituted for the proof in § 166.

Proposition XV. Theorem

168. *If two straight lines drawn from a point in a perpendicular to a given line cut off equal segments from the foot of the perpendicular, they are equal.*



Given: $RF \perp AB$, P any point in RF, and PD and PE drawn from P to AB, making $FD = FE$.

To prove: $PD = PE$.

Proof:

Left to the pupil.

169. Corollary. *If two straight lines drawn from a point in a perpendicular to a given line cut off unequal segments from the foot of the perpendicular, the more remote is the greater.*

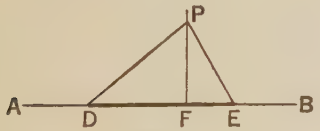


Fig. 1

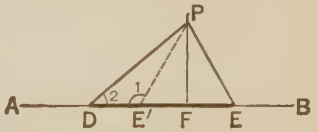


Fig. 2

Given: $DF > EF$.

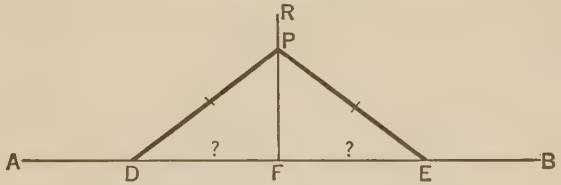
To prove: $DP > EP$.

Analysis:

1. How can we get a line equal to EP? (See § 168.)
2. In terms of this line (Fig. 2) what is to be proved?
3. What kind of angle is $\angle 1$? Why? $\angle 2$? Why?
4. Hence, compare DP and $E'P$; DP and EP. Explain.

Proposition XVI. Theorem

170. *If two straight lines drawn from a point in a perpendicular to a given line are equal, they cut off equal segments from the foot of the perpendicular.* (Converse of Th. XV.)



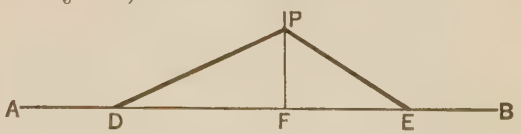
Given: $RF \perp AB$ and $PD = PE$.

To prove: $DF = EF$.

Proof:

Left to the pupil.

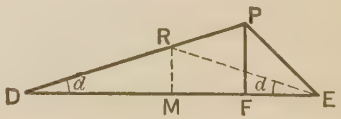
171. Corollary. *If two straight lines drawn from a point in a perpendicular to a given line are unequal, the greater cuts off the greater segment from the foot of the perpendicular.* (Converse of § 169.)



Analysis:

1. In the drawing what is given? What is to be proved?
2. Show that the hypothesis would be contradicted if (*a*) $DF = EF$ or (*b*) $DF < EF$. What is the conclusion?

Ex. Prove § 171, using the following drawing and outline:



Show: 1. $DF > DM$.

2. $DF > ME > FE$.

Is the cons. implied in the drawing permissible? Explain.

Proposition XVII. Theorem

172. *If two triangles have two sides of one equal respectively to two sides of the other and the included angles unequal, the triangle which has the greater included angle has the greater third side.*

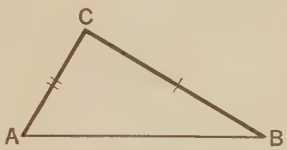


Fig. 1

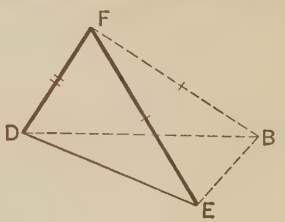


Fig. 2

Given: $\triangle ABC$ and DEF, with $AC = DF$, $BC = EF$, and $\angle C > \angle DFE$.

To prove: $AB > DE$.

Analysis:

1. $\triangle ABC$ may be placed in the position DBF (Fig. 2). Then what line inequality is to be proved?
2. What angle inequality would prove this? Why? (See § 166.)
3. What angle equality is known in the drawing? (See § 116.)
4. Hence show how step 2 may be obtained by Ax. 9.

Proof:

Left to the pupil.

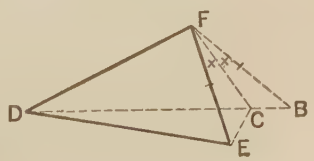
Discussion:

When would E fall on DB? within the triangle?

Ex. Prove Th. XVII, using the following construction and outline for Fig. 2.

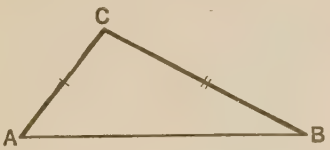
Show: 1. $EC = BC$.
2. $DE < (DC + CE)$ or $(DC + CB)$.

This proof may be used instead of the one given above.



Proposition XVIII. Theorem

173. *If two triangles have two sides of one equal respectively to two sides of the other and the third sides unequal, the triangle which has the greater third side has the greater angle opposite.* (Converse of Th. XVII.)



Given: $\triangle ABC$ and DEF, with $AC = DF$, $BC = EF$, and $AB > DE$.

To prove: $\angle C > \angle F$.

Analysis:

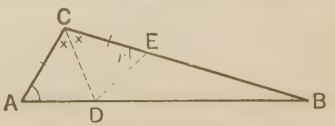
1. $\angle C = \angle F$ or $\angle C < \angle F$ or $\angle C > \angle F$. Which is to be proved true?
2. If $\angle C = \angle F$, compare the $\triangle$. Compare AB and DE. Does this contradict the hypothesis?
3. If $\angle C < \angle F$, compare AB and DE. (See § 172.)
4. Hence what is the conclusion?

Proof:

Left to the pupil.

Exercises — Group 31

1. Prove Th. XIII, p. 85, using the following drawing and outline:



Given: $\triangle ABC$, $CB > AC$, CD bisector of $\angle C$, and $CE = CA$.

Prove: $\angle A\ (= \angle r) > \angle B$.

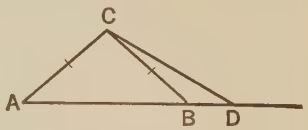
This proof may be used instead of the one given in the text.

2. Prove Th. XIII, p. 85, by extending CA to E, making $CE = CB$.

3. What is the longest side in a right triangle? in an obtuse triangle? Why?

4. If an isosceles triangle is obtuse, which is the longest side?

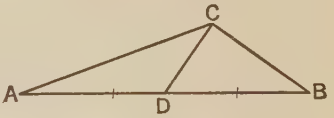
5. In the isosceles triangle ABC ($CA = CB$), AB is extended any distance to D and D joined to C. Prove that $\angle DAC > \angle D$.



6. Given: As in ex. 5.
Prove: $DC > BC$ or AC.

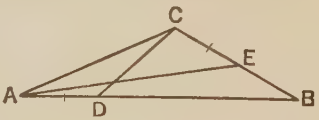
7. In ex. 6 if D is any point on AB between A and B, prove $DC < AC$ or BC.

8. Given: $AC > BC$ and D the mid-point of AB.
Prove: $\angle CDA > \angle CDB$.



9. $\triangle ABC$ is isosceles ($AC = BC$) and D is any point in BC.
Prove: $AD > DB$.

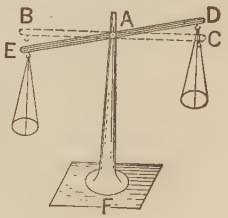
10. Given: $AB > BC$ and $AD = CE$.
Prove: $AE > CD$.



11. In the isosceles triangle ABC ($CA = CB$), CA is extended its own length to D.
Prove: $DB > CB$.

12. Given: $AD = BC$ and $AC > DB$.
Prove: $\angle ADC > \angle DCB$.

13. When the scale pans of an ordinary balance are not on the same level, show that each pan is nearer the middle post than when the pans are on the same level. Show that the pans are always at equal distances from the middle post, no matter how much the bar is tipped.



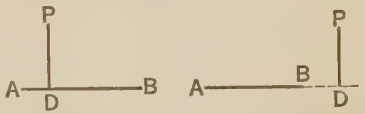
DISTANCE

174. Since distance as applied to geometric figures is always measured along the shortest path, the following statements are at once apparent:

1. *The distance between two points* is the length of the straight line joining the two points. (See Post. 2.)

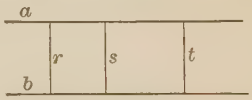
2. *The distance from a point to a line* is the length of the perpendicular from the point to the line. (See § 167.)

Thus, in the drawing the distance from P to AB is the length of the perpendicular PD.



3. *The distance between two parallel lines* is the length of the perpendicular drawn to one line from any point in the other.

Thus, in the drawing the distance between the two parallel lines a and b is the length of any of the lines r, s, t.



175. *An altitude of a triangle* is the perpendicular drawn from any vertex to the opposite side, which is then considered as the *base*.

Hence the distance from any vertex of a triangle to the opposite side is the length of the altitude drawn from that vertex.

Exercises — Group 32

1. Could any meaning be attached to the statement "the distance between any two lines"? Explain.

2. Show that in an isosceles triangle each end of the base is the same distance from the opposite side.

3. Show that any point in the bisector of an angle is the same distance from the sides of the angle.

4. Show that any point in the perpendicular bisector of a line is the same distance from either end of the line.

***5.** Show that two parallel lines are everywhere the same distance apart. (Use § 157.)

QUADRILATERALS — PARALLELOGRAMS

176. A *quadrilateral* is a polygon of four sides.

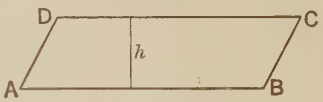
The parts are defined as in any polygon. (See § 57.)

177. A *parallelogram* is a quadrilateral whose opposite sides are parallel.

Any side of a parallelogram may be regarded as its *base*.

The *altitude* of a parallelogram is the line perpendicular to the base from any point in the opposite side.

Thus, $ABCD$ is a parallelogram, frequently designated as $\square AC$, AB is a base, and h is the corresponding altitude.



Can h be dropped from any point in DC? Why?

178. The angles of a parallelogram may be oblique or right. Unless otherwise stated, the former will be understood.

179. A *rhombus* is a parallelogram having two adjacent sides equal.

It will be shown later that all the sides of a rhombus are equal.

180. A *rectangle* is a parallelogram one of whose angles is a right angle.

Show that all the angles of a rectangle are right angles.

181. A *square* is a rectangle having two adjacent sides equal.

It will be shown later that all the sides of a square are equal.

These various figures are illustrated in the following drawings:

Proposition XIX. Theorem

182. *The opposite sides of a parallelogram are equal, and the opposite angles are equal.*

Given: $\square ABCD$.

To prove: $AB = DC$, $AD = BC$, $\angle A = \angle C$, and $\angle ABC = \angle ADC$.

Analysis:

1. What method of proving lines and $\measuredangle$ = is suggested?
2. What cons. will make it possible to use this method?
3. Mention with reasons the parts that are = in the two $\triangle$.
4. Are the $\triangle \cong$? Are the parts in question = ? Why?

Proof:

Left to the pupil.

183. Corollary 1. *All the sides of a rhombus are equal, and all the sides of a square are equal.*

184. Corollary 2. *A diagonal divides a parallelogram into two congruent triangles.*

185. Corollary 3. *Parallel lines included between parallel lines are equal.*

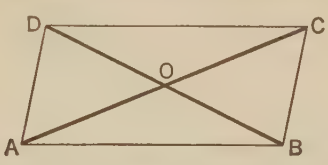
186. Corollary 4. *Two parallel lines are everywhere the same distance apart.*

Discussion:

1. What two statements in the proof of § 182, when combined, prove $\angle ABC = \angle ADC$? Explain.
2. What cons. would have made these angles corr. $\measuredangle$ of $\cong \triangle$?

Proposition XX. Theorem

187. *The diagonals of a parallelogram bisect each other.*



Given: $\square ABCD$ with diagonals AC and BD intersecting at O.

To prove: $AO = OC$, $BO = OD$.

Proof:

Left to the pupil.

Ex. 1. In § 182, prove $\angle A = \angle C$ by extending AB to E and showing that $\angle A = \angle EBC = \angle C$.

Ex. 2. In § 182, prove $\angle A = \angle C$ by using § 148.

Ex. 3. If the opposite angles of a parallelogram are supplementary, the parallelogram is a rectangle.

Ex. 4. Lines through the vertices of any quadrilateral parallel to the diagonals form a parallelogram equal to twice the quadrilateral.

Ex. 5. In § 187, if $\angle COD = 130°$ and $\angle OAD = 50°$, find the value of all the angles that you can.

Ex. 6. How many pairs of congruent triangles are there in the drawing of § 187?

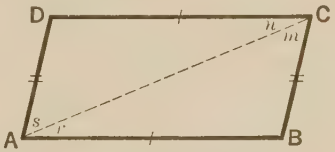
Ex. 7. In the drawing of § 187, pick out eight pairs of angles that are equal.

Ex. 8. If the median to the hypotenuse AB of the right triangle ABC is extended its own length to D and DA and DB are drawn, prove that $ADBC$ is a rectangle.

* **Ex. 9.** Prove that any two consecutive angles of a parallelogram are supplementary.

Proposition XXI. Theorem

188. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given: Quad. $ABCD$ with $AB = DC$ and $AD = BC$.
To prove: $ABCD$ is a $\square$.

Analysis:

1. By def. what must be shown to make $ABCD$ a $\square$?
2. In terms of the drawing state what must be proved to make this relation true.
3. What means will probably be used?

Proof: Left to the pupil.

Proposition XXII. Theorem

189. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

Given: Quad. $ABCD$ with $AB = DC$ and $AB \parallel DC$.
To prove: $ABCD$ is a $\square$.

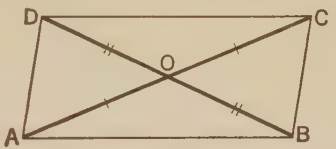
Analysis:

See outline in Th. XXI.

Proof: Left to the pupil.

Proposition XXIII. Theorem

190. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Given: Quad. $ABCD$ with diagonals AC and BD intersecting at O and with $AO = OC$, $BO = OD$.

To prove: $ABCD$ is a $\square$.

Analysis:

1. By Th. XXII what must be shown to make $ABCD$ a $\square$?
2. State this in terms of the drawing.

Proof: Left to the pupil.

Ex. 1. Of what theorem is Th. XXI the converse?

Ex. 2. Prove Th. XXII by means of § 188.

Ex. 3. In the isosceles $\triangle ABC$ ($AC = BC$), AC is extended its own length to D and a line through $D \parallel BA$ meets BC extended in E. Prove that $ABDE$ is a parallelogram.

Ex. 4. Of what theorem is Th. XXIII the converse?

Ex. 5. Prove Th. XXIII, basing the proof on Th. XXI.

Ex. 6. Prove that if the mid-points of the adjacent sides of a parallelogram are joined in order, another parallelogram is formed.

Ex. 7. Prove that $\square ABCD$ is equilateral if the diagonal AC bisects $\angle A$.

Ex. 8. $ABCD$ is a quadrilateral. If the altitudes on AC from D and B are equal, then AC bisects BD.

Exercises — Group 33

1. In the figure, how should the gate $ABCD$ be braced to prevent sagging? Which would be better, a brace from A to C or from B to D? Why?

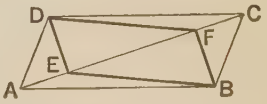
2. $ABCD$ is a parallelogram with E and F the mid-points of AB and CD respectively. Prove that $AECF$ is a parallelogram.

***3.** Any line through O, the intersection of the diagonals of a parallelogram, terminated by the sides is bisected at O.

Here O is called the *center of symmetry* of the parallelogram.

4. Given: $\square ABCD$, DE and BF each $\perp AC$.

Prove: $EBFD$ is a parallelogram. (See § 189.)



***5.** Prove that the diagonals of a rectangle are equal.

***6.** If the diagonals of a parallelogram are equal, the figure is a rectangle. (Converse of ex. 5.)

7. Show how a door frame or a window frame may be "squared" by measuring and comparing the distances between opposite corners.

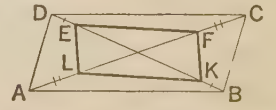
8. Given: $\square ABCD$, with $DE = BK$.

Prove: $AKCE$ is a parallelogram.

9. Prove ex. 8 when E and K are taken at equal distances from D and B respectively on DB extended through the vertices.

10. Given: $\square ABCD$, with $AL = CF$, $DE = BK$.

Prove: $LKFE$ is a parallelogram.



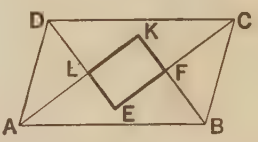
11. Prove ex. 10 when L, K, F, E are taken on the diagonals extended through the vertices.

* **12.** Prove that the diagonals of a rhombus bisect the angles of the rhombus and are perpendicular to each other.

13. E is any point in the diagonal DB of $\square AC$ and EK and EJ are drawn parallel to BC and BA respectively. Prove that $EKDJ$ is a parallelogram.

14. Given: $\square AC$ with AK, BK, CE, and DE bisecting $\measuredangle A$, B, C, D respectively.

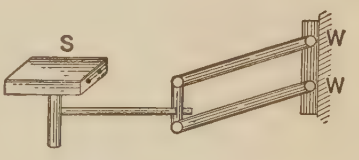
Prove: $EFKL$ is a rectangle. (See ex. 9, p. 95.)



* **15.** A quadrilateral is a parallelogram if (*a*) three consecutive angles taken in order are supplementary in pairs, (*b*) the opposite angles are equal in pairs.

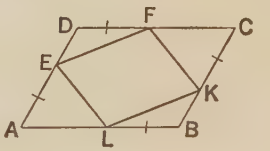
* **16.** Two parallelograms are congruent if they have two sides and the included angle equal respectively.

17. The figure shows an adjustable bracket fastened to the wall at W and supporting a shelf at S. Show that S always remains horizontal as it is raised or lowered.



18. Given: $\square AC$, with $DF = CK = BL = AE$.

Prove: $LKFE$ is a parallelogram.



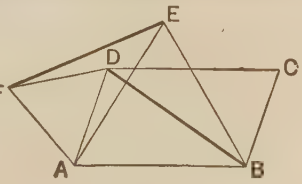
19. Given: AB, CD, and EF, three equal and parallel lines, and ACE, a straight line.

Prove: $\triangle ADE \cong \triangle BCF$.

20. Given: $\square ABCD$ and equilateral $\triangle FAD$ and EAB.

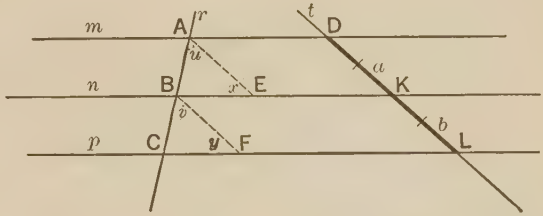
Prove: $FE = DB$.

Sug. See § 110.



Proposition XXIV. Theorem

191. *If three or more parallels cut off equal segments on one transversal, they cut off equal segments on any other transversal.*



Given: The three ∥s, m, n, p, cutting off equal parts a, b on a transversal t, and cutting off AB, BC on any other transversal r.

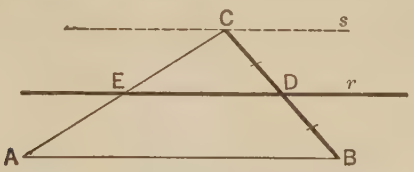
To prove: $AB = BC$.

Analysis:

1. The plan is to form △ containing these lines and to prove the △ ≅.
2. A and B are the probable points from which to draw lines forming these △.
3. The hypothesis and the figure would suggest drawing the lines on the same side of r and ∥ t.

Proof: Statements	Reasons
1. Draw AE and $BF \parallel t$.	1. § 132
2. $\therefore AE \parallel BF$.	2. § 133
3. $\therefore \angle u = \angle v$,	3. § 142
4. Also $\angle x = \angle y$.	4. § 148
5. $AEKD$ and $BFLK$ are ▱.	5. § 177
6. Hence $AE = a$ and $BF = b$.	6. § 182
7. $\therefore AE = BF$.	7. Ax. 1
8. Hence $\triangle ABE \cong \triangle BCF$.	8. § 113
9. $\therefore AB = BC$.	9. § 65

192. COROLLARY. *The line parallel to one side of a triangle and bisecting a second side bisects the third side.*



Given: $r \parallel AB$, and $CD = DB$.
To prove: $CE = EA$.

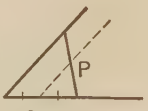
Analysis:

1. In the drawing what corresponds to t of the figure in § 191? to r? a? b? AB? BC? p? n?
2. How can we get a line to correspond to m?

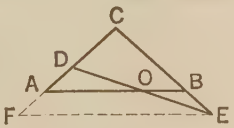
Discussion:

1. In the theorem must $AB = a$? Explain.
2. Must the ‖s be the same distance apart? Prove your statement.
3. Suppose $r \parallel t$. Discuss.
4. Suppose the transversals crossed between the ‖s. Would $ABFE$ be a $\square$? Explain.
5. Show by drawings two positions which r and t may have that would make $AB = a$, etc. Prove your statement.
6. Show that the part cut off on each ‖ differs from the next part by the same amount.

Ex. 1. Show how to draw a line through a given point within a given angle, terminating in the sides of the angle and bisected by the point.

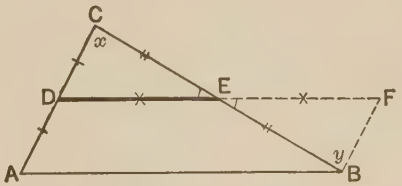


Ex. 2. Given: $AC = BC$ and $AD = BE$.
Prove: $DO = OE$.



Proposition XXV. Theorem

193. *The line that joins the mid-points of two sides of a triangle is parallel to the third side and equals half the third side.*



Given: $\triangle ABC$, with D and E the mid-points of AC and BC respectively.

To prove: (*a*) $DE \parallel AB$; (*b*) $DE = \frac{1}{2} AB$.

Analysis:

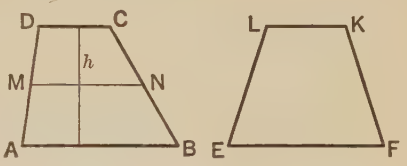
1. If DE, extended its own length to F, formed a $\square ABFD$, both (*a*) and (*b*) would be proved. Explain.
2. What relation must exist between AD and BF to make $ABFD$ a $\square$? (See § 189.)
3. What $\measuredangle$ must be proved $=$ to make $AD \parallel BF$?
4. What method is suggested? Does this help to prove $AD = BF$? Explain.

Proof:

For both (*a*) and (*b*) left to the pupil.

194. A *trapezoid* is a quadrilateral with two and only two sides parallel. The parallel sides are called the *bases* and the non-parallel sides are called the *legs* of the trapezoid. The line joining the mid-points of the legs of a trapezoid is called the *median* of the trapezoid. A line from any point in one base perpendicular to the other base is called the *altitude* of the trapezoid.

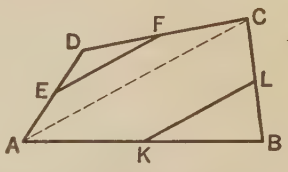
Thus, in the drawing $ABCD$ is a trapezoid, AB and DC are the bases, AD and BC are the legs, MN is the median, and h is the altitude.



195. If the non-parallel sides of a trapezoid are equal, the trapezoid is called an *isosceles trapezoid*.

Thus, in the drawing above, $EFKL$ is an isosceles trapezoid.

Exercises — Group 34



* **1.** Given: Quad. $ABCD$, with K, L, F, E the mid-points of the sides in order.

Prove: (*a*) $EF = KL$ and $EF \parallel KL$.

(*b*) $EFLK$ is a parallelogram.

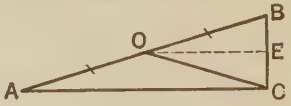
* **2.** In ex. 1 what kind of parallelogram is formed if $ABCD$ is a rhombus? a rectangle? a square? Prove your statements.

3. If the mid-points of the three sides of a triangle are joined, the triangle is divided into four congruent triangles.

* **4.** The mid-point of the hypotenuse of any right triangle is the same distance from the three vertices.

Prove: $AO = BO = CO$.

Sug. Draw $OE \perp BC$.



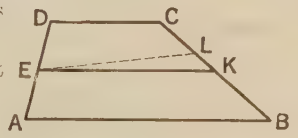
5. A line parallel to the bases of a trapezoid, bisecting one of the legs, bisects the other leg also.

Sug. Use § 191.

* **6.** The median of a trapezoid is parallel to the bases.

Given: Trap. $ABCD$, E the mid-point of AD, and K the mid-point of BC.

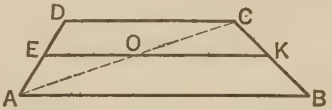
Prove: $EK \parallel AB \parallel DC$.



Sug. Use ex. 5 and the indirect method.

* **7.** The median of a trapezoid bisects the two diagonals.

* **8.** The median of a trapezoid is equal to one half the sum of the two bases.



Show: $EK = \frac{1}{2}(AB + DC)$.

9. In an isosceles trapezoid the two legs make equal angles with either base.

Sug. See § 162.

10. The diagonals of an isosceles trapezoid are equal.

11. In an isosceles trapezoid show that the triangle formed by the diagonals and either base is isosceles.

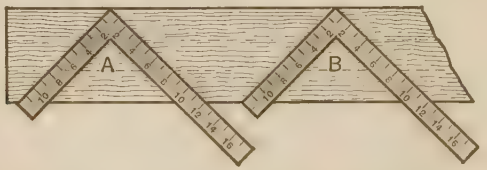
12. If the medians are drawn to the equal sides of an isosceles triangle, the line joining their end points is parallel to the base.

* **13.** The line joining the mid-points of the two diagonals of a trapezoid is parallel to the bases and equals one half their difference.

Sug. Use the indirect method.

14. If the mid-points of the legs of an isosceles triangle are joined to the mid-point of the base, a rhombus is formed.

15. A plank is to be sawed into three strips of equal width. Place the square as in *A* and make marks at the 4″ and 8″ di-



visions. Repeat the operation with the square similarly placed at *B*. Lines joining corresponding marks will be parallel and divide the plank into three equal parts. Why?

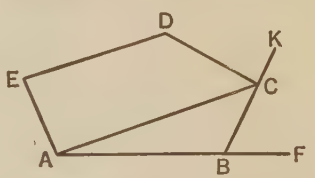
POLYGONS

196. A *polygon* is a plane figure bounded by three or more straight lines. (See § 57.)

The points of intersection of the lines are the *vertices* of the polygon, and the segments of these lines between adjacent vertices are the *sides* of the polygon.

Any angle formed by two consecutive sides of the polygon is called an *interior angle* of the polygon, or, more briefly, an *angle* of the polygon. An angle formed by one side and the prolongation of the adjacent side is called an *exterior angle* of the polygon. (See §§ 46–47.)

In the drawing, $ABCD\ldots$ is a polygon, A, B, $C\ldots$ are the vertices, AB, BC, . . . are the sides, $\angle EAB$, $\angle ABC$, $\angle BCD$, $\angle D$, . . . are interior angles, $\angle FBC$, $\angle KCD$, . . . are exterior angles, and AC is a diagonal.



197. 1. *In any polygon the sum of the exterior angle and the adjacent interior angle at each vertex is a straight angle.*

Thus, $\angle FBC + \angle ABC =$ a st. $\angle$. Hence,

2. *In any polygon having* ***n*** *vertices, that is,* ***n*** *sides, the sum of all the exterior angles and all the interior angles of the polygon is* ***n*** *straight angles.*

198. Polygons Classified as to Sides. Polygons are usually named from the number of sides. A polygon is called a *triangle* if it has three sides; a *quadrilateral*, if it has four sides; a *pentagon*, if it has five sides; a *hexagon*, if it has six sides. The next names in order are *heptagon*, *octagon*, *nonagon*, *decagon*, *undecagon*, *duodecagon*.

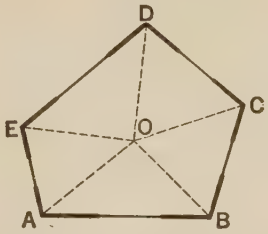
An *equilateral polygon* is one having all its sides equal.

199. Polygons Classified as to Angles. A polygon is *equiangular* if all its angles are equal.

A polygon is *convex* if each of its interior angles is less than a straight angle. In this text, unless otherwise stated, *polygon* means *convex polygon*.

Proposition XXVI. Theorem

200. *The sum of the angles of a polygon of* ***n*** *sides is* $(n-2)$ *straight angles.*



Given: Polygon ABC . . . having n sides.
To prove: Sum of the $\measuredangle$ of ABC . . . is $(n-2)$ st. $\measuredangle$.

Analysis:

1. Upon what theorem will this proof probably depend?
2. If any point within the polygon is joined to each vertex, $\triangle$ will be formed containing all the $\measuredangle$ of the polygon.
3. How many $\triangle$ will there be? Hence what will be the sum of all the $\measuredangle$ of the $\triangle$?
4. What $\measuredangle$ in this sum are not in the required sum?
5. How many st. $\measuredangle$ are there in the sum of these $\measuredangle$? Why?
6. Hence how many st. $\measuredangle$ are there in the required sum?

Proof:

Left to the pupil.

201. Corollary 1. *In an equiangular polygon of* ***n*** *sides, each angle equals* $\frac{n-2}{n}$ *straight angles.*

Analysis:

1. What will be the sum of all the $\measuredangle$?
2. How many $\measuredangle$ will there be? How do they compare?

202. COROLLARY 2. *The sum of the exterior angles of a polygon, made by producing each of its sides in succession, is equal to two straight angles.*

Analysis:

1. In a polygon of n sides what does ext. $\measuredangle$ + int. $\measuredangle$ =, by § 197?
2. What does int. $\measuredangle$ + $\measuredangle$ at O = in the drawing? (Step 3 of analysis.)
3. Hence show that ext. $\measuredangle$ + int. $\measuredangle$ = int. $\measuredangle$ + $\measuredangle$ at O (Ax. 1).
4. $\therefore$ ext. $\measuredangle$ = $\measuredangle$ at O = 2 st. $\measuredangle$ (Ax. 3).

Discussion:

1. If O were taken in one side (but not a vertex), how many $\triangle$ would be formed? What would be the sum of all the $\measuredangle$ of the $\triangle$? What $\measuredangle$ would be subtracted from this sum to get the required sum? Does this give $(n - 2)$ st. $\measuredangle$?
2. If O were one of the vertices of the polygon, how many $\triangle$ would be formed? What would be the sum of all the $\measuredangle$ of the $\triangle$? Must any $\measuredangle$ be subtracted to get the required sum? Is the sum $(n - 2)$ st. $\measuredangle$, as before?

Ex. 1. Find the sum of the angles of a hexagon; of a heptagon; of an octagon.

Ex. 2. If four of the angles of a pentagon are 94°, 112°, 110°, and 98°, what is the fifth angle?

Ex. 3. If two angles of a quadrilateral are 70° and 110°, how many degrees are there in the sum of the other two angles?

Ex. 4. How many degrees are there in each angle of an equiangular polygon of 4 sides? of 5 sides? 8 sides? 10 sides?

Ex. 5. How many sides has an equiangular polygon if one of its angles is 120°? 150°?

Ex. 6. Using the formula in § 201, show that the smallest angle of any equiangular polygon is 60°. What is this polygon?

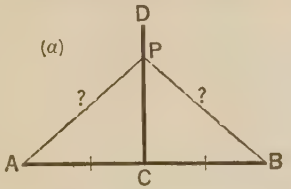
Ex. 7. How many sides has an equiangular polygon if one of its exterior angles is $25\frac{5}{7}°$?

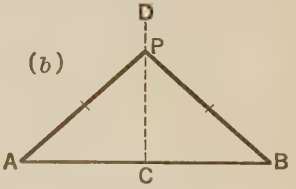
LOCATION THEOREMS

Proposition XXVII. Theorem

203. (*a*) *Every point in the perpendicular bisector of a line is equidistant from the ends of that line;* and

(*b*) Conversely, *every point equidistant from the ends of a line lies in the perpendicular bisector of that line.*





(*a*) **Given**: *P any* point in *DC*, the ⊥ bisector of *AB*.
To prove: *AP* = *BP*.

(*b*) **Given**: Line *AB* and point *P* such that *AP* = *BP*.
To prove: *P* is on the ⊥ bisector of *AB*.

Proof:

For both (*a*) and (*b*) left to the pupil.

(*b*) **Analysis**:

1. How should the line through *P* to *AB* be drawn? (See § 118.)
2. In terms of this figure, what remains to be shown of this line?

204. Corollary. *Two points each equidistant from the ends of a line determine the perpendicular bisector of that line.*

Use § 203 (*b*) and § 10.

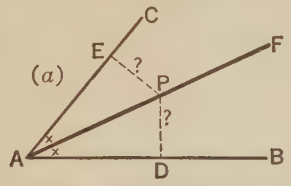
Discussion:

1. Where in the proof of (*a*) is it shown that *every* point in the ⊥ bisector of the line has been considered?
2. Where in the proof of (*b*) is it shown that *every* point equidistant from the ends of the line has been considered?
3. Show that there are two different ways in which *PC* may be drawn in (*b*) and in each case state what fact remains to be proved to make *PC* the ⊥ bisector of *AB*.

Proposition XXVIII. Theorem

205. (*a*) *Every point in the bisector of an angle is equidistant from the sides of that angle;* and

(*b*) Conversely, *every point equidistant from the sides of an angle lies in the bisector of that angle.*



(*a*) **Given**: $\angle BAC$, bisector AF, and P *any* point in AF.
To prove: P is equidistant from AB and AC.

(*b*) **Given**: $\angle BAC$ and P *any* point in the angle equidistant from AB and AC; that is, $PD = PE$.
To prove: P lies in the bisector of $\angle BAC$.

(*a*) **Analysis**:

1. How are these distances obtained?
2. Hence what lines are to be proved equal?

Proof:

For both (*a*) and (*b*) left to the pupil.

(*b*) **Analysis**:

1. How is AF drawn?
2. State, in terms of $\measuredangle$ of the drawing, what is to be proved.

Discussion:

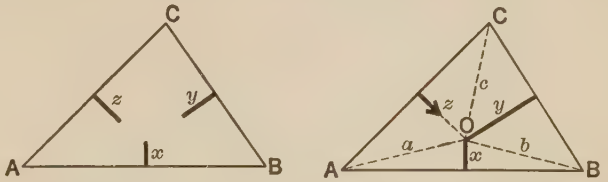
1. Where in the proof of (*a*) is it shown that *every* point in the bisector of the angle has been considered?
2. Where in the proof of (*b*) is it shown that *every* point equidistant from the sides of the angle has been considered?
3. Why should there be two different ways in which PC may be drawn in § 203 (*b*) and only one way in § 205 (*b*)?

CONCURRENT LINE THEOREMS

206. Three or more lines which meet in a point are said to be *concurrent.*

Proposition XXIX. Theorem

207. *The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the three vertices.*



Given: $\triangle ABC$ with x, y, z $\perp$ bisectors of AB, BC, CA respectively.

To prove: (*a*) x, y, z meet in a point.

(*b*) This point is equidistant from A, B, C.

(*a*) **Analysis**:

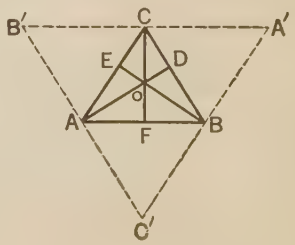
1. Must any two of these lines, as x and y, meet? Why?
2. For this point to lie on z what condition must it satisfy? Then what two lines must be proved equal?
3. To what line is each one equal? Why?
4. Then does z pass through O? Why?

Proof:

For both (*a*) and (*b*) left to the pupil.

(*b*) **Analysis**:

Is O equidistant from A, B, C? Why?

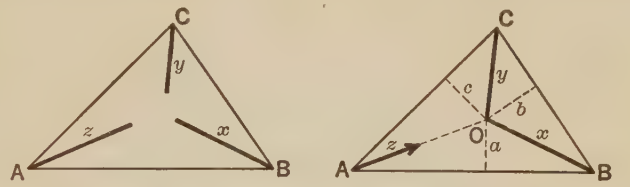


208. Corollary. *The three altitudes of a triangle meet in a point.*

Use ex. 15, p. 99, and § 207 (*a*).

Proposition XXX. Theorem

209. *The bisectors of the angles of a triangle meet in a point which is equidistant from the three sides.*



Given: $\triangle ABC$ with x, y, z the bisectors of $\measuredangle B$, C, A respectively.

To prove: (a) x, y, z meet in a point.

(b) This point is equidistant from AB, BC, and CA.

Analysis:

Use the outline given in Th. XXIX with the proper change in (b).

Proof:

For both (a) and (b) left to the pupil.

Ex. 1. Where is O in § 207 if ABC is a right triangle? an obtuse triangle?

Ex. 2. Must O in § 209 always lie *within* the triangle? Explain.

***Ex. 3.** Can a point be found equidistant from *any* three points?

Ex. 4. Can a point be found equidistant from *any* three lines?

***Ex. 5.** In $\triangle ABC$, show that the bisectors of the interior angle B and the exterior angles at C and A meet in a point which is equidistant from the three sides.

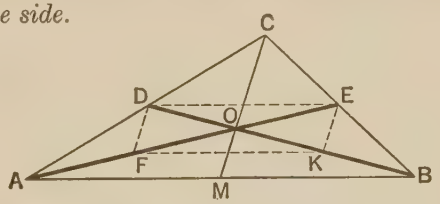
This is an *excenter* of the triangle. A triangle has three excenters.

Ex. 6. Where is the point in which the three altitudes of a triangle meet when the triangle is right? obtuse? Illustrate.

210. A *median* of a triangle is the line drawn from any vertex to the mid-point of the opposite side.

Proposition XXXI. Theorem

211. *The medians of a triangle meet in a point which is two thirds the distance from each vertex to the mid-point of the opposite side.*



Given: $\triangle ABC$ with medians AE, BD, CM.

To prove: (*a*) AE, BD, and CM meet, as at O.

(*b*) $AO = \frac{2}{3}AE$, $BO = \frac{2}{3}BD$, $CO = \frac{2}{3}CM$.

(*a*) **Analysis:**

Why must *any* two medians, as AE and BD, meet?

Proof:

For both (*a*) and (*b*) left to the pupil.

(*b*) **Analysis:**

1. If F is the mid-point of AO and K is the mid-point of BO, show that $AO = \frac{2}{3}AE$ and $BO = \frac{2}{3}BD$ if $FO = OE$ and $KO = OD$.
2. Show that these lines are = if $FKED$ is a $\square$.
3. Show that $FKED$ is a $\square$ by §§ 193 and 189.
4. Where would median CM cross AE? Why?
5. What conclusion follows?

212. The point in which the three perpendicular bisectors of the sides of a triangle meet is called the *circumcenter* of the triangle. (See § 207.) If a circle is described with O as center, and a, b, or c as radius, it will pass through A, B, and C. The circle is then said to be *circumscribed about the triangle* and the triangle is said to be *inscribed in the circle.*

213. The point in which the three altitudes of a triangle meet is called the *orthocenter* of the triangle. (See § 208.)

The word is derived from the Greek word *ortho* meaning *straight* or *upright.*

214. The point in which the three bisectors of the angles of a triangle meet is called the *incenter* of the triangle. (See § 209.)

If a circle is described with O as center and a, b, or c as radius, it will touch the sides AB, BC, and CA. The circle is then said to be *inscribed in the triangle* and the triangle is said to be *circumscribed about the circle.*

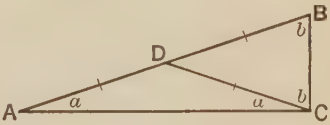
215. The point in which the three medians of a triangle meet is called the *centroid* of the triangle. (See § 211.) The centroid (meaning literally *center-like*) is the center of mass or the center of gravity of the triangle.

Exercises — Group 35

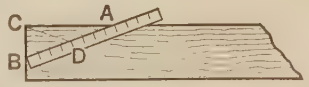
*** 1.** Prove that in an equilateral triangle the three perpendicular bisectors of the sides are also the three bisectors of the angles, the three medians, and the three altitudes; and that all these lines are equal.

*** 2.** Prove that the median to the hypotenuse of a right triangle is equal to one half the hypotenuse. (This is merely a restatement of ex. 4, p. 103.)

*** 3.** Prove that if the median to a side of a triangle is equal to one half that side, the opposite angle is a right angle. (Converse of ex. 2.)



4. In the drawing, show how it is possible to determine with the aid of a ruler whether the corner at C is square. Explain.



5. Two equilateral triangles are congruent if an altitude of one equals an altitude of the other.

6. The angle between the median and the altitude drawn to the hypotenuse of a right triangle equals the difference between the two acute angles. (See ex. 2 above; also ex. 7, p. 83.)

216. Summary of the Ways of Proving Various Relations between Triangles, Lines, Angles, etc.

I. *Any two triangles are congruent:*
1. If they have one side and any two angles equal respectively.
2. If they have two sides and the included angle equal respectively.
3. If they have the three sides equal respectively.

II. *Any two right triangles are congruent:*
1. If they have the two legs equal respectively.
2. If they have one leg and either acute angle equal respectively.
3. If they have the hypotenuse and either leg equal respectively.
4. If they have the hypotenuse and either acute angle equal respectively.

III. *Any two angles are equal:*
1. If they are straight angles, right angles, vertical angles.
2. If they are complements or supplements of the same angle or of equal angles.
3. If they are corresponding angles of congruent triangles.
4. If they are angles opposite equal sides in the *same* triangle.
5. If they are alternate interior or corresponding angles of parallel lines made by the *same* transversal.
6. If they are angles whose sides are parallel or perpendicular respectively.
7. By the use of certain axioms. Which ones?

IV. *Any two lines are equal:*
1. If they are corresponding sides of congruent triangles.
2. If they are sides opposite equal angles in the *same* triangle.
3. If they are opposite sides of a parallelogram.
4. If they are segments between parallels which cut equal segments from another transversal.
5. By the use of certain axioms. Which ones?

V. *Any two lines are parallel:*
   1. If two alternate interior angles or two corresponding angles are equal when the lines are cut by the *same* transversal.
   2. If they are parallel or perpendicular to a third line.
   3. If they are opposite sides of a parallelogram.
   4. If the sum of the interior angles on the same side of a *common* transversal equals a straight angle.

VI. *Any two angles are unequal:*
   1. If they lie opposite unequal sides in the *same* triangle.
   2. If one is an exterior angle and the other is an opposite interior angle of the *same* triangle.
   3. By Prop. XVIII and certain axioms. Which ones?

VII. *Any two lines are unequal:*
   1. If they lie opposite unequal angles in the *same* triangle.
   2. If when drawn from a point in a perpendicular to a given line they cut off on the given line unequal distances from the foot of the perpendicular.
   3. By Prop. XVII and certain axioms. Which ones?

VIII. *Any quadrilateral is a parallelogram:*
   1. If the opposite sides are parallel.
   2. If the opposite sides are equal.
   3. If two sides are equal and parallel.
   4. If the diagonals bisect each other.
   5. If the opposite angles are equal or the adjacent angles are supplementary.

217. Ways in Which a Construction Line May Be Drawn.

A line may be drawn:

1. Joining two points.
2. Bisecting a given angle.
3. Through a point perpendicular to a given line.
4. Through a point parallel to a given line.
5. Extending a given line its own length.
6. Marking off the shorter of two lines on the longer.
7. Making with another line an angle equal to a given angle.

218. Analytic Method of Proof. This method of proof traces out a path from the *unknown* statement made in the conclusion to some *known* statement, by a series of successive substitutions. It argues that a given proposition is true if another proposition is true, continuing this substitution until some known statement is reached. The purpose of every step is therefore evident. It is the method which should invariably be used in discovering proofs in geometry.

219. Synthetic Method of Proof. This method consists in continually putting together *known* statements to obtain new statements until eventually the *unknown* statement made in the conclusion is obtained. Its procedure is therefore the reverse of that used in the analytic method. While the truth of successive steps taken in this method is always evident, nevertheless the steps follow more or less blindly and the pupil often wonders for what purpose they are taken. This is the method used in most texts in geometry for recording proofs of propositions.

We may therefore say that the analytic method should be the method used in *discovering* proofs in geometry, while the synthetic method should be the method used in *recording* or *presenting* the proofs *after they have been discovered.*

For illustrations of these methods see §§ 130, 152, etc.

220. A General Method of Attack in Discovering Proofs of Exercises. (See § 129.)

1. *Make an accurate drawing to represent what is given.*
2. *State specifically in terms of lines and angles of the drawing what is given and what is to be proved.*
3. *Recall the various ways of proving this required relation of parts.*
4. *Determine the most probable method to be used in this particular exercise.*
5. *Note what is known and what is lacking to apply this method.*
6. *Study how to get the parts that are lacking, either by a reapplication of the above steps or by some construction line.*

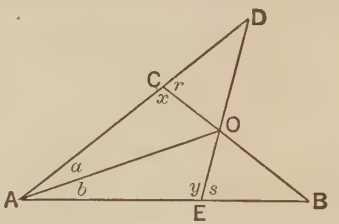
7. *When a direct method of proof is not apparent, try the indirect method of proof.*

8. *When two magnitudes are given unequal, marking off the less on the greater often suggests a line of proof.*

In following out any suggested line of proof, do not abandon the method until the work has been carried to a point where the method is shown to be impossible or where there is no apparent way of proceeding farther along that line. Then go back and start another possible line of attack.

221. The following exercise illustrates the above method of attack.

Ex. ABC is a triangle in which $AB > AC$. D is taken on AC produced so that $AD = AB$, and E is taken on AB so that $AE = AC$. If DE is drawn to cut BC at O, show that AO will bisect $\angle A$.



Given: $\triangle ABC$ with $AB > AC$; also $AD = AB$, $AE = AC$, and lines DE and AO drawn.

Prove: $\angle a = \angle b$.

Analysis:

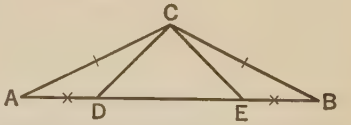
1. The most probable method is $\cong \triangleq$, a possible pair being $\triangle EAO$ and $\triangle CAO$.
2. These $\triangleq$ have $AC = AE$ and $AO = AO$. Could both pairs of lines be used in s.a.s. combination? Explain.
3. How many parts would be lacking for a.s.a.?
4. If $AC = AE$ is used in s.a.s. combination, what $\measuredangle$ and sides are required?
5. In what other $\triangleq$ do the required $\measuredangle$ occur? Are these $\cong \triangleq$?
6. In what other $\triangleq$ do the required sides occur? Can they be proved $\cong$ by hyp. and step 5? Explain.
7. Then are the $\triangleq$ in step 1 $\cong$? Does $\angle a = \angle b$?

Proof: Left to the pupil.

Exercises — Group 36

Congruent Triangles

1. If $AC = BC$ and $AD = EB$, prove: (1) $\triangle ADC \cong \triangle BEC$. (2) $\triangle AEC \cong \triangle BDC$.

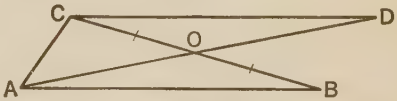


2. A line is drawn through the intersection of the two diagonals of a parallelogram, ending in two parallel sides. Name pairs of congruent triangles and prove.

3. If $AF = EB$, $AD = BC$, and $DE = FC$, prove: $\triangle AED \cong \triangle BFC$.

4. In ex. 3, if lines DF and CE are drawn, prove: (1) $\triangle AFD \cong \triangle BEC$; (2) $\triangle DFE \cong \triangle CEF$.

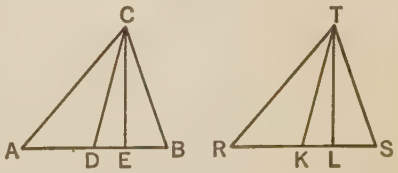
5. Given: O mid-point of BC and $CD \parallel AB$.
Prove: $\triangle ABO \cong \triangle CDO$.



6. If equal distances are marked off from the base on the equal sides of an isosceles triangle and these points are joined to the opposite ends of the base, prove that three pairs of congruent triangles are formed.

7. Two triangles are congruent if they have two sides and the altitude on one of these sides equal respectively.

***8.** In the figure, $AB = RS$, median CD = median TK, and altitude CE = altitude TL.
Prove: $\triangle ABC \cong \triangle RST$.
Sug. Start with § 162.



9. Two right triangles are congruent if the hypotenuses and the altitudes on the hypotenuses are equal respectively.

Sug. Use ex. 2, p. 113.

Exercises — Group 37

Lines and Angles Equal

1. If the legs of an isosceles trapezoid are extended to meet, they form with either base an isosceles triangle.

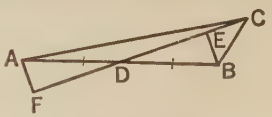
2. If two exterior angles of a triangle are equal, the triangle is isosceles.

3. The bisectors of the base angles of an isosceles triangle are equal.

4. Two equal lines AC and AD are drawn on opposite sides of AB and make equal angles with AB. Show that BC and BD will also make equal angles with AB.

5. Given: $\triangle ABC$, median CD, and AF and BE each $\perp CD$.

Prove: $AF = BE$.

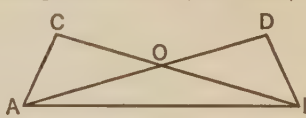


6. The bisectors of the base angles of an isosceles triangle form with the base an isosceles triangle.

7. If the bisector of an angle of a parallelogram is extended to meet the opposite side, an isosceles triangle is formed (two cases).

8. Given: $AC = BD$ and $AD = BC$.

Prove: $\triangle ABO$ is isosceles.



9. If the bisector of the angle between two sides of a triangle is perpendicular to the third side, the first two sides are equal.

Of what statement is this the converse?

10. Two points in the base of an isosceles triangle equally distant from the ends of the base are equally distant from the vertex.

11. Perpendiculars to the base of an isosceles triangle at equal distances from the ends of the base cut off equal distances on the legs.

12. If the mid-points of the sides of an isosceles triangle are joined, another isosceles triangle is formed.

13. If in trapezoid $ABCD$ base $AB = AD$ and $\angle A = 2\ \angle C$, prove that $AB = \frac{1}{2}\ CD$.

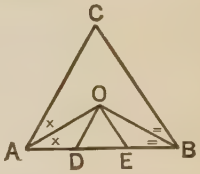
14. Given: $\square AC$ and $DE \parallel BF$.
Prove: $DE = BF$.

15. If two altitudes of a triangle are equal, the triangle is isosceles.

Of what statement is this the converse?

16. If the upper base of an isosceles trapezoid equals one of the legs, the diagonals bisect the angles at the lower base.

17. Given: Equilateral $\triangle ABC$, with AO and BO bisectors of $\angle A$ and $\angle B$ respectively, $OD \parallel CA$, and $OE \parallel CB$.
Prove: $AD = DE = EB$.



18. The altitudes upon a diagonal of a parallelogram from two opposite vertices are equal.

19. The segments of the bisectors of two opposite angles of a parallelogram cut off by the diagonal are equal.

222. Inequality Axioms.

(*a*) Axiom 10. *If equals are added to, or subtracted from, unequals, the results are unequal in the same order.*

Thus, if $a > b$ and $c = d$, then $(a + c) > (b + d)$ and $(a - c) > (b - d)$.

(*b*) Axiom 11. *If unequals are multiplied or divided by equals, the results are unequal in the same order.*

Thus, if $a < b$ and $c = d$, then $(a \times c) < (b \times d)$ and $\frac{a}{c} < \frac{b}{d}$.

(*c*) Axiom 12. *If unequals are added to unequals of the same order, the sums are unequal in the same order.*

Thus, if $a > b$ and $c > d$, then $(a + c) > (b + d)$.

(*d*) Axiom 13. *If unequals are subtracted from equals, the remainders are unequal in the reverse order.*

Thus, if $a = b$ and $c < d$, then $(a - c) > (b - d)$.

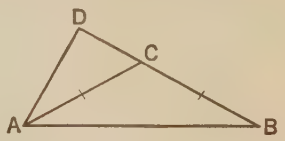
Exercises — Group 38

Lines and Angles Unequal

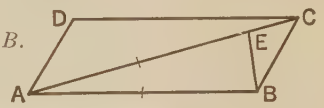
*1. Any side of a triangle is less than the sum of the other two sides.

*2. Any side of a triangle is greater than the difference between the other two sides.

3. Given: $AC = BC$ and BC extended any distance to D.
Prove: $AD < BD$.

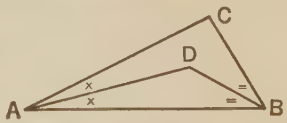


4. Given: $\square ABCD$ and $AE = AB$.
Prove: $BC > EC$.



5. In triangle ABC the bisector of angle A meets BC in D.
Prove: $AB > BD$ and $AC > CD$.

6. Given: $AC > BC$, AD bisector of $\angle A$, and BD bisector of $\angle B$
Prove: $AD > BD$.



7. If two adjacent sides of a parallelogram are unequal, any pair of adjacent angles formed by the diagonals are unequal. (See § 173.)

8. State and prove the converse of ex. 7.

9. If any point within a triangle is joined to the three vertices, the sum of the three lines is greater than one half the perimeter.

10. The sum of the sides of any quadrilateral is less than twice the sum of its diagonals.

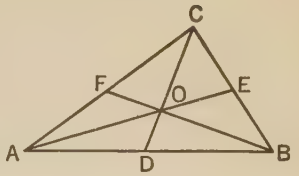
11. The sum of the sides of any quadrilateral is greater than the sum of the diagonals.
Sug. Start with one of the large triangles.

12. Given: $\triangle ABC$, CD a median, and $AC > BC$.
Prove: $\angle DCB > \angle DCA$.
Sug. Extend CD its own length to E and join E and A.

13. Given: $\triangle ABC$, and medians AE, BF, and CD intersecting in O.

Prove: $2(AE + BF + CD) > (AB + BC + CA)$.

Sug. Start with $\triangle AOD$. See § 211.



14. Prove that four times the sum of the three medians of a triangle is greater than three times the sum of the sides.

Sug. In the drawing of ex. 13, start with $\triangle AOB$.

15. Prove Th. XIV, p. 86, using the following outline:

(*a*) Draw a line from B to E on AC produced, so that $\angle ABE = \angle BAE$.

(*b*) Show that $(CB + CE) > BE$ or AE or $(AC + CE)$.

(*c*) Then use Ax. 10, p. 120.

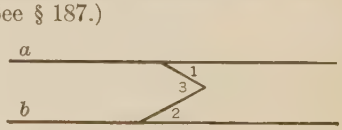
Exercises — Group 39

Use of Construction Lines

1. The lines joining the mid-points of the opposite sides in any quadrilateral bisect each other. (See § 187.)

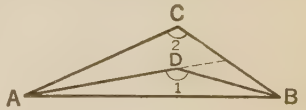
2. Given: $a \parallel b$.

Prove: $\angle 1 + \angle 2 = \angle 3$.



3. If a and b are two parallels and c is a parallel midway between a and b, prove that if any transversal cuts a and b, the segment between a and b is bisected by c.

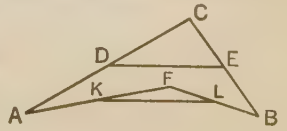
* **4.** In the drawing, prove: $\angle 1 > \angle 2$.



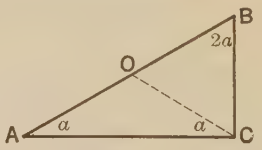
5. Find a point in one side of a triangle equidistant from the other two sides.

6. Given: K, L, E, D mid-points of AF, BF, BC, AC respectively.

Prove: (*a*) $DE \parallel KL$; (*b*) $DE = KL$.



***7.** If one acute angle of a right triangle is double the other, that is, one 30° and the other 60°, the hypotenuse is double the shorter leg.



Given: $\angle B = 2 \times \angle A$.

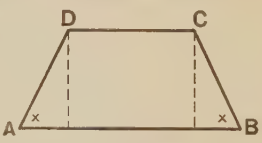
Prove: $AB = 2 \times BC$.

Sug. Make $\angle ACO = \angle A$.

***8.** If the hypotenuse of a right triangle is double the shorter leg, one acute angle is 30° and the other is 60°. (Converse of ex. 7.)

Sug. See ex. 2, p. 113.

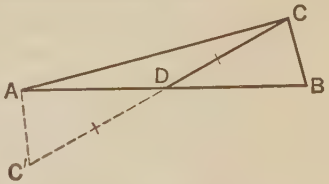
9. Given: Trapezoid $ABCD$, with $\angle A = \angle B$.



Prove: $ABCD$ is isosceles; that is, $AD = BC$.

10. The base AB of triangle ABC is divided into four equal parts by the points D, E, F. Show that the line from D to the center of AC equals the line from F to the center of BC.

***11.** The median to any side of a triangle is less than one half the sum of the other two sides.

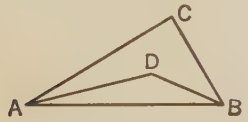


Given: $\triangle ABC$ and median CD.

Prove: $CD < \frac{1}{2}(AC + BC)$; that is, $2\,CD < (AC + BC)$.

***12.** If two lines are drawn to the ends of a given line from any point outside the line, their sum is greater than the sum of any other pair of lines, drawn in the same way, but included by the first pair.

Prove: $(AC + BC) > (AD + BD)$.



Analysis:

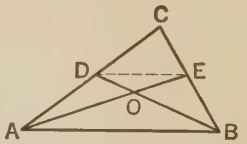
1. Complete the inequality: $(AC + CE) > ?$
2. Complete the inequality: $(DE + EB) > ?$
3. Add the two inequalities.

Could BD have been extended to AC? Explain.

Exercises — Group 40

INDIRECT METHOD

1. Two lines drawn within a triangle from two vertices to the opposite sides cannot bisect each other.



Given: $\triangle ABC$, with AE and BD drawn from A and B to BC and AC.

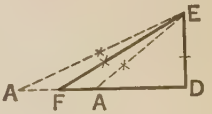
Prove: AE and BD cannot bisect each other. (See § 190.)

2. In ex. 1, if $\angle C$ is obtuse, show that $\angle AOB$ cannot be a right angle.

3. Prove Th. XII, p. 84, by superposition.

Given: Rt. $\triangle ABC$ and FDE, with $BC = DE$ and $AC = FE$.

Prove: $\triangle ABC \cong \triangle FDE$.



Analysis:

1. Lay $\triangle ABC$ on $\triangle FDE$, BC coinciding with DE and BA lying on DF. Why?
2. A must fall to the left of F or on F or to the right of F.
3. Which do we wish to prove true? Would this make the $\triangle \cong$?
4. Show that each of the other two possibilities leads to contradictions. (See § 169.) What conclusion follows?

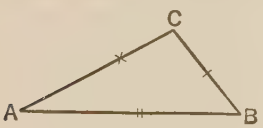


FIG. 1

FIG. 2

4. Prove Th. IV, p. 60, by superposition.

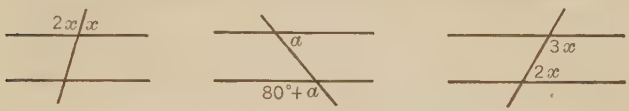
Analysis:

1. Place $\triangle ABC$ on $\triangle DEF$, A on D and B on E.
2. Then C must fall on F or on some other point.
3. Which do we wish to prove true? Would this make the $\triangle \cong$?
4. If C fell as in Fig. 2, then the $\perp$ bisector of CF would pass through D and E. (See ex. 6, p. 67.)
5. Is this possible? Why? What conclusion follows?

Exercises — Group 41

NUMERICAL

1. Find all the angles in each of the following drawings:



2. One angle of an isosceles triangle is 40°. What are the angles of the triangle? Give two possible sets of angles.

3. The vertex angle of an isosceles triangle is 48°. Find (*a*) an exterior angle at the base; (*b*) the angle between the bisectors of the base angles; (*c*) the angle between the altitudes on the legs; (*d*) the angle between the base and the altitude on one of the legs.

4. The bisector of one of the equal angles of an isosceles triangle forms with the opposite side an angle of 108°. Find all the angles.

5. The diagonals of a rectangle form with each other an angle of 130°. Find the angles which they form with the sides.

6. Two diagonals of a parallelogram form with each other an angle of 118°. At one end of the base a diagonal forms angles of 26° and 34° with the sides. Find all the angles of the figure.

7. One angle of a rhombus is 120° and the shorter diagonal is 12″. What is the length of one side of the rhombus?

8. One exterior angle at the base of an isosceles trapezoid is $1\frac{1}{3}$ right angles. Find all the angles of the trapezoid.

9. If two sides of a triangle are 12″ and 8″, between what values must the third side be?

10. The length of the hypotenuse of a right triangle is 18. Find the length of the median to the hypotenuse.

11. The sides of a triangle are 12″, 18″, and 22″. Find the sides of the triangle formed by joining the mid-points of the three sides.

12. The bases of a trapezoid are 8″ and 20″. Find the length of (*a*) the median; (*b*) the line joining the mid-points of the diagonals.

13. How many sides has an equiangular polygon if the sum of its exterior angles is equal to the sum of its interior angles?

Exercises — Group 42

MISCELLANEOUS

* **1.** Lines through the vertices of a triangle parallel to the opposite sides form a triangle whose sides are twice the sides of the given triangle.

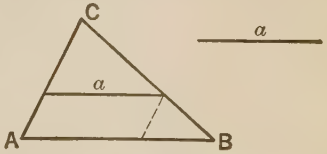
2. $ABCD$ is a parallelogram and E and F are points in AB and CD respectively such that $DE = DA$ and $BF = BC$. Prove that $EBFD$ is a parallelogram.

3. Lines drawn from the mid-point of the base of an isosceles triangle to the mid-points of the legs form with the legs a rhombus or a square. When is it a rhombus? when a square?

4. The angle between the bisectors of two angles of a triangle equals half the third angle plus a right angle.

* **5.** If through any point in the bisector of an angle lines are drawn parallel to the sides of the angle, these lines form with the sides of the angle a rhombus.

* **6.** (*a*) Show how to draw a line of given length, a, ending in two sides of a triangle and parallel to the third side. (*b*) Suppose $a > AB$.



* **7.** If one of the equal sides of an isosceles triangle is extended through the vertex its own length, the line joining the end point to the nearer end of the base is perpendicular to the base.

* **8.** If one angle of a triangle is equal to the sum of the other two angles: (*a*) the triangle can be divided into two isosceles triangles; (*b*) the first angle is a right angle.

* **9.** Show how §§ 72, 75, and 76 each can be proved by § 204.

10. If from any point inside an angle perpendiculars are drawn to the two sides, the angle formed between the two perpendiculars is the supplement of the given angle.

11. If two sides of a triangle are unequal, the median to the third side makes unequal angles with that side.

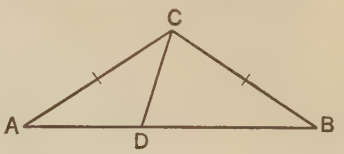
12. The lines joining the mid-points of the sides of an isosceles trapezoid taken in order form a rhombus or a square.

*** 13.** The two lines joining the feet of two altitudes of a triangle to the mid-point of the third side are equal. (See ex. 2, p. 113.)

14. The line joining the feet of two altitudes of a triangle is bisected by the perpendicular to this line from the mid-point of the third side. (See § 163.)

15. Given: $AC = BC$ and $AD < DB$.

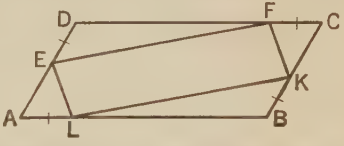
Prove: $\angle DCB > \angle ACD$.



16. If M and N cut off equal distances from A and C on the sides AB and CD of the parallelogram $ABCD$, prove that $AMCN$ is a parallelogram.

17. Given: $\square ABCD$, also $ED = FC = KB = LA$.

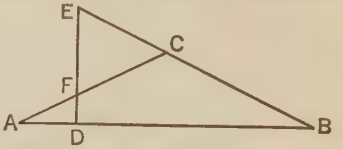
Prove: $LKFE$ is a parallelogram.



18. The mid-points of two opposite sides of any quadrilateral and the mid-points of the diagonals are the vertices of a parallelogram. (See § 193.)

19. Given: Isosceles $\triangle ABC$, with $AC = BC$ and $ED \perp AB$.

Prove: $\triangle CEF$ is isosceles.

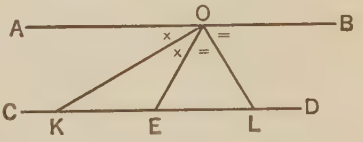


*** 20.** (*a*) If the diagonals of a trapezoid are equal, the trapezoid is isosceles. (Use §§ 162 and 110.)

(*b*) State the converse of (*a*).

21. Given: $AB \parallel CD$, OK bisecting $\angle EOA$, and OL bisecting $\angle EOB$.

Prove: $KE = EO = EL$.



22. ABC is a triangle in which $\angle A > \angle B$ and P is any point in the median CD. If P is joined to A and B, prove that $\angle PAB > \angle PBA$.

23. (*a*) If two medians of a triangle are equal, the triangle is isosceles. (See § 211.)

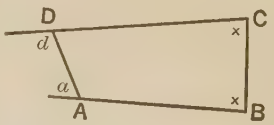
(*b*) State the converse of (*a*).

24. In the isosceles triangle ABC ($CA = CB$), CB is extended through B to D and AD is drawn.

Prove: $AD > BD$.

25. Given: $\angle B = \angle C$, $AB < DC$.

Prove: $\angle a < \angle d$.



Sug. Extend BA and CD to meet.

Suppose $\angle B$ and $\angle C$ were right angles.

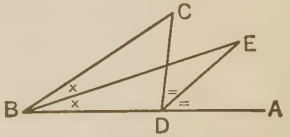
26. $ABCD$ is a quadrilateral in which AB is the longest side and DC is the shortest side.

Prove: (*a*) $\angle C > \angle A$; (*b*) $\angle D > \angle B$.

27. Show how to draw a line parallel to the base of a triangle cutting the other two sides so that the sum of the segments adjacent to the base shall equal the parallel line. (See ex. 7, p. 76.)

28. Given: BE bisector of $\angle ABC$ and DE bisector of $\angle ADC$.

Prove: $\angle E = \frac{1}{2} \angle C$.



29. The lines from two opposite vertices of a parallelogram to the mid-points of two parallel sides trisect the diagonal joining the other two vertices.

30. Any angle of a triangle is acute, right, or obtuse, according as the median from it is greater than, equal to, or less than, one half the opposite side.

31. D and E are the feet of perpendiculars from A to the bisectors of angles B and C respectively of triangle ABC.

Prove: $DE \parallel BC$.

Sug. Extend AD and AE to meet BC and use § 193.

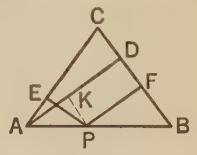
32. D, E, F are the mid-points of sides AB, BC, CA respectively of triangle ABC, and K is the foot of the perpendicular from C to AB.

Prove: $\angle EDF = \angle C = \angle EKF$.

Sug. Show that the parts of $\angle C$ and $\angle EKF$ are equal.

33. The bisectors of the angles of a rectangle form a square.

*** 34.** The sum of the perpendiculars from any point in the base of an isosceles triangle to the equal sides is equal to the altitude on one of the equal sides.



Prove: $PE + PF = AD$; that is, $PE + PF = AK + KD$.

How is PK drawn?

35. In the triangle ABC, E is the mid-point of the median from A and BE extended meets AC in F.

Prove: $FC = 2 \times AF$; that is, $AF = \frac{1}{3}\, AC$.

36. In the triangle ABC, medians AD and CE are extended their own lengths to points P and F respectively and these points are joined to B.

Prove: (*a*) FBP is a straight line; (*b*) $ED = \frac{1}{4}\, FP$.

Sug. Show that the angles at B equal the angles of $\triangle ABC$ respectively.

37. Given: $\triangle ABC$, with CD bisecting $\angle C$, $AF \perp CD$, and E the mid-point of AB.

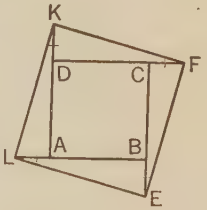
Prove: $EF = \frac{1}{2}\,(BC - AC)$.

Sug. Produce AF to meet BC at K.

38. Given: Triangle ABC, with equilateral triangles ABC', BCA', and CAB' constructed on the three sides.

Prove: $AA' = BB' = CC'$.

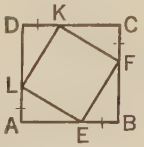
Sug. Use § 110.



39. Given: Square $ABCD$ and $EB = FC = KD = LA$.

Prove: $EFKL$ is a square.

Give proof for both drawings.

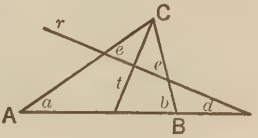


40. Given: t bisector of $\angle C$ and $r \perp t$.

Prove: (1) $\angle e = \frac{1}{2}\,(\angle b + \angle a)$.

(2) $\angle d = \frac{1}{2}\,(\angle b - \angle a)$.

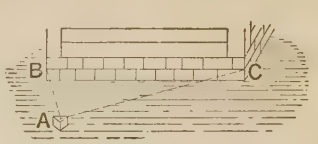
Sug. Use § 153.



Exercises — Group 43

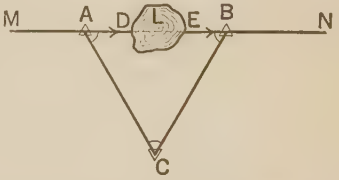
Applications

1. Two distances are measured from a tack A in a stake to the corners B and C of a near-by building. Show that if the stake is destroyed, another may be easily reset.

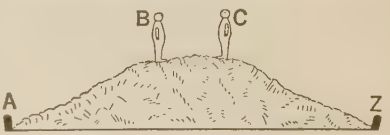


2. In attempting to locate a swarm of bees in a woods a man observes a bee at work on a flower until it is filled with honey and then carefully notes the direction in which it flies. He then locates another bee some distance away and makes similar observations. How does he use this information in locating the swarm?

3. To produce a straight line MD through an obstacle L, a distance AC is set off at 60° to MD and a distance $CB = AC$ is set off at 60° to AC. Then EB, set off at 60° to CB, will be the continuation of MD. Why?



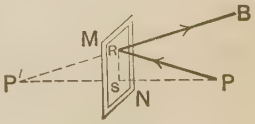
4. To run a straight line over a hill from A to Z, two observers, B and C, take places on the ridge as nearly as they can guess in a line between the two points, so that each can see the other and the point beyond him. B looks at Z and by signal puts C "in line"; C then looks at A and puts B in line. This is repeated until B–C–Z and C–B–A are simultaneously in line. Show that A–B–C–Z are then all in line.



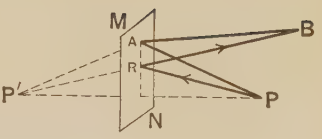
5. What geometric principle does a carpenter use in testing the perpendicularity of a board with a plumb line?

6. A ray of light starting at P is reflected by a plane mirror MN to point B. Locate point R, where the ray strikes the mirror.

Sug. Locate P', then R, and construct the path PRB. (See ex. 13, p. 68.)



7. PRB is the path of a ray reflected by a plane mirror MN from P to B. Show that the path PRB is shorter than the path PAB, A being any other point on MN.



Sug. $P'B < (P'A + AB)$, etc.

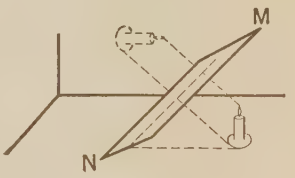
This proves that light is reflected along the shortest path.

8. A straight road is inclined 30° to the horizontal and a man is walking along the road at the rate of $2\frac{1}{2}$ miles an hour. How many feet does he ascend vertically against the force of gravity in 15 minutes?

9. Explain why the extension gate shown in the drawing at the right moves horizontally when pulled out.

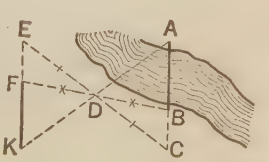
10. In ex. 1, p. 67, if AC and BC are not equal, point M may be determined as follows: Place the feet A and B on any two points and mark on DE where the plumb line crosses. Turn the instrument, end for end, resting it on the same points, and mark the new crossing of the plumb line. The point on DE midway between the two crossings is the correct location of M. Explain.

11. A plane mirror MN is inclined to the floor of a room at an angle of 45°. Show that the image of any object placed vertically before it on the floor is horizontal.



12. To find the distance between two points A and B *when only one point is accessible.* (See ex. 10, p. 57.)

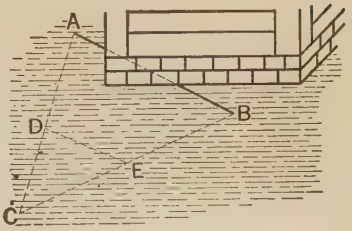
Assume C anywhere in line with the required distance AB. Assume D anywhere, make $DE = DC$ and $DF = DB$. Establish K in line AD and at the same time in line EF. Then $KF = AB$. Explain.



13. To determine the distance between two points A and B separated by a building the following measurements were made:

$AD = 40'$, $AC = 80'$, $CB = 100'$, $CE = 50'$, $DE = 36'$.

Show how to find the distance AB from these measurements.



14. An *optical square* is a small hand instrument for laying off a right angle. It consists essentially of two vertical mirrors a and a' set at an angle of 45° and two openings o and o' directly above the mirrors (see Fig. 1 below).

The lines to the instrument from two objects P and P' are at right angles when P' can be seen in mirror a', by reflection from a, in line with P observed through opening o'. Explain. (See ex. 13, p. 68.)

Fig. 1

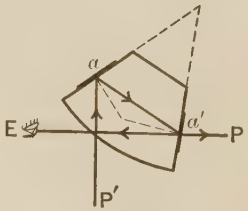
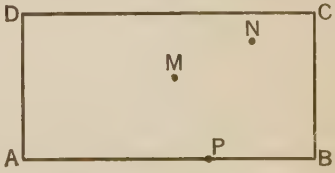


Fig. 2

15. When a billiard ball strikes a cushion it is reflected according to the same law by which a ray of light is reflected from a plane mirror. (See ex. 13, p. 68.)

At what point P must the ball M strike the cushion AB to be reflected so as to strike ball N?



16. Ball M is driven to cushion AB, reflected to cushion BC, and finally hits ball N. Construct the path of the ball.

BOOK TWO

THE CIRCLE

223. From the definitions found in §§ 60–63, the following properties of circles may be inferred or easily proved:

1. In the same circle or in equal circles, all radii are equal and all diameters are equal.

2. Circles having equal radii or equal diameters are equal.

3. If the centers of two equal circles which lie in the same plane are made to coincide, the circles will coincide.

4. A point is within a circle, on a circle, or without a circle, according as its distance from the center is less than, equal to, or greater than, the radius.

5. A straight line through any point within a circle cuts the circle at two points.

6. With a given point as center and a given line as radius, one and only one circle can be drawn; that is, a circle is determined by these parts. (See §§ 10 and 68.)

224. *Equal arcs,* in the same circle or in equal circles, are arcs that may be made to coincide throughout. Hence:

1. If the extremities of two arcs of the same circle or of equal circles can be made to coincide, the arcs are equal; for all the other points must coincide.

2. If two arcs of a circle are equal, their extremities can be made to coincide.

Exercises — Group 44

1. Describe the method used in making two equal circles coincide and in making two equal arcs of the same circle coincide. Wherein do the two methods differ?

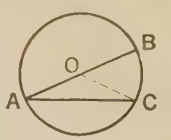
2. Mention three positions a point or a line may have with reference (*a*) to the center of the circle; (*b*) to the circle.

3. If the radius of a circle is 3″, describe the position, with reference to the circle, of a point (*a*) 5″ from the center; (*b*) $2\frac{1}{2}$″ from the center; (*c*) 3″ from the center.

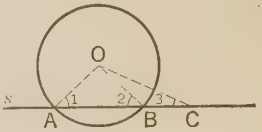
4. If two arcs of a circle are made to coincide, would the chords of those arcs coincide? Explain.

5. A diameter of a circle is greater than any other chord.

In the drawing at the right, prove: $AB > AC$.



6. A straight line cannot cut a circle in more than two points.



Analysis:

1. Suppose s cuts O in three points, A, B, and C. Draw OA, OB, and OC.
2. Compare OA, OB, and OC. Compare $\angle 1$ and $\angle 2$; $\angle 1$ and $\angle 3$; then $\angle 2$ and $\angle 3$.
3. Show that this relation between $\angle 2$ and $\angle 3$ is impossible. (See § 121.)

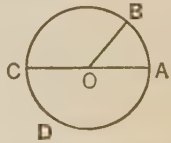
7. A bullet is fired horizontally into a circular post, the bullet passing completely through the post. Show that there will be two and only two holes made by the bullet in the surface of the post.

225. A *minor arc* is an arc less than a semicircle.

226. A *major arc* is an arc greater than a semicircle.

227. A *central angle* is an angle formed by two radii.

Thus, in the drawing, $\overset{\frown}{AB}$ is a minor arc, $\overset{\frown}{ADB}$ is a major arc, and $\angle AOB$ is a central angle of circle O.



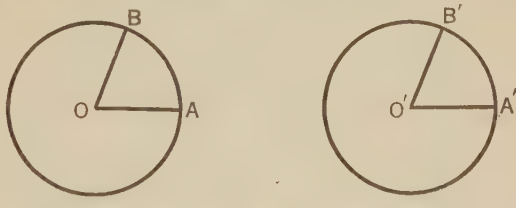
228. An angle is said to *intercept* the arc cut off between its sides, and the arc is said to *subtend* or *determine* the angle.

Thus, in the above drawing, $\angle AOB$ intercepts $\overset{\frown}{AB}$ and $\overset{\frown}{AB}$ subtends or determines $\angle AOB$.

PROPOSITION I. THEOREM

229. (*a*) *If in the same circle or in equal circles two central angles are equal, the arcs which they intercept are equal;* and

(*b*) Conversely, *if in the same circle or in equal circles two arcs are equal, the central angles which they subtend are equal.*



(*a*) **Given**: Equal circles O and O', with $\angle AOB = \angle A'O'B'$.

To prove: $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

(*b*) **Given**: Equal circles O and O', with $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

To prove: $\angle AOB = \angle A'O'B'$.

(*a*) **Analysis**:

1. Use superposition. Explain in terms of the drawing.
2. What lines match? What points coincide? Why?
3. Are the arcs equal? Why? (See § 224, 1.)

Proof:

For both (*a*) and (*b*) left to the pupil.

(*b*) **Analysis**:

1. Use superposition. Explain the method.
2. What points coincide? What lines match? Why?

230. COROLLARY. (*a*) *If in the same circle or in equal circles two central angles are unequal, the greater angle intercepts the greater arc;* and

(*b*) Conversely, *if in the same circle or in equal circles two arcs are unequal, the greater arc subtends the greater central angle.*

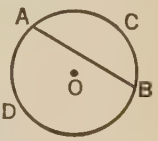
Ex. 1. Any diameter of a circle bisects the circle.

SUG. What are the two central angles formed?

Ex. 2. Any two intersecting diameters intercept two pairs of equal arcs.

Ex. 3. Two quadrants of a circle subtend equal central angles.

231. A chord is said to *subtend* or *determine* an arc. In all cases, except when the arc is a diameter, the two arcs will be unequal. (Why?) One arc will be a minor arc, the other, a major arc. If neither is specified, the minor arc is meant.



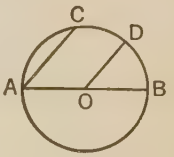
Thus, in the drawing, chord AB subtends or determines the minor arc ACB and the major arc ADB.

Exercises — Group 45

1. Given: Circle O and diameter AO; also $\angle 1 = \angle 2$.

Prove: $\overset{\frown}{AC} = \overset{\frown}{AD}$.

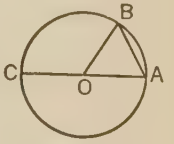
2. If from the mid-point of the arc intercepted between two radii perpendiculars are drawn to the two radii, these perpendiculars are equal.



3. Given: Diameter AB and chord AC; also radius $OD \parallel AC$.

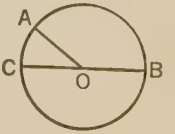
Prove: $\overset{\frown}{BD} = \overset{\frown}{DC}$.

4. Two diameters perpendicular to each other divide a circle into four equal parts.



5. Given: Circle O, diameter AC, and chord AB = a radius.

Prove: $\overset{\frown}{CB} > \overset{\frown}{AB}$.



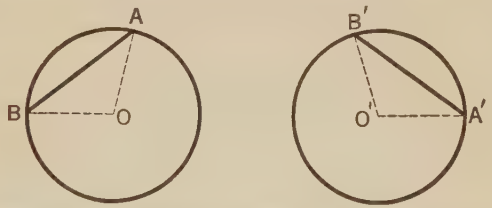
6. Given: Circle O, diameter BC, and $\overset{\frown}{AB}$ > a quadrant.

Prove: $\angle BOA > \angle AOC$.

Proposition II. Theorem

232. (*a*) *If in the same circle or in equal circles two chords are equal, the arcs subtended by them are equal;* and

(*b*) Conversely, *if in the same circle or in equal circles two arcs are equal, the chords subtended by them are equal.*



(*a*) **Given**: Equal circles O and O', with chord AB = chord $A'B'$.

To prove: $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

(*b*) **Given**: Equal circles O and O', with $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

To prove: Chord AB = chord $A'B'$.

(*a*) **Analysis**:

1. What condition makes two arcs equal?
2. Hence in the drawing what must be proved?
3. What method is suggested?

Proof:

For both (*a*) and (*b*) left to the pupil.

(*b*) **Analysis**:

What method of proving lines equal is suggested?

Ex. 1. If two chords bisect each other, prove: (*a*) the opposite intercepted arcs are equal; (*b*) the two chords are diameters.

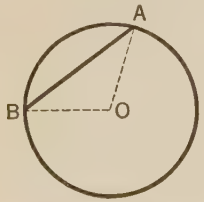
Ex. 2. Arcs AB, BC, CD, and DA are quadrants of circle O. Show that $ABCD$ is an equilateral quadrilateral.

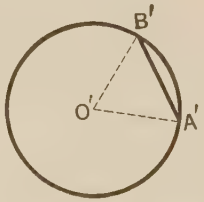
Ex. 3. AOB is a diameter of circle O and BC and BD are equal chords drawn on opposite sides of AB. Show that chord AC = chord AD.

Proposition III. Theorem

233. (*a*) *If in the same circle or in equal circles two chords are unequal, the greater chord subtends the greater minor arc.*

(*b*) Conversely, *if in the same circle or in equal circles two minor arcs are unequal, the greater arc is subtended by the greater chord.*





(*a*) **Given**: Equal circles O and O', with chord $AB >$ chord $A'B'$.

To prove: $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.

(*b*) **Given**: Equal circles O and O', with $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.

To prove: Chord $AB >$ chord $A'B'$.

(*a*) **Analysis**:

1. What condition in the figure would make $\overset{\frown}{AB} > \overset{\frown}{A'B'}$?
2. Show by § 173 that this condition is true.

Proof:

For both (*a*) and (*b*) left to the pupil.

(*b*) **Analysis**:

1. What condition in the two $\triangle$ would make ch. $AB >$ ch. $A'B'$?
2. Is this condition true? Explain.

Ex. 1. In the inscribed triangle ABC, $\angle C > \angle A$.

Prove: $\overset{\frown}{AB} > \overset{\frown}{BC}$.

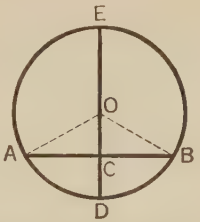
Ex. 2. (*a*) AB is a diameter of a circle and BC and BD are two chords on the same side of AB such that $BC > BD$.

Prove: Chord $AC <$ chord AD. (See Ax. 13, p. 120.)

(*b*) Suppose that BC and BD are on opposite sides of AB.

PROPOSITION IV. THEOREM

234. *A diameter perpendicular to a chord bisects the chord and the arcs subtended by it.*



Given: Circle O, chord AB, and diameter $ED \perp AB$.

To prove: $AC = BC$, $\overset{\frown}{AD} = \overset{\frown}{BD}$, and $\overset{\frown}{AE} = \overset{\frown}{BE}$.

Analysis:

1. What method is suggested for proving $AC = BC$?
2. Under what conditions in the drawing would $\overset{\frown}{AD} = \overset{\frown}{BD}$? Is this condition true? Explain.
3. Under what condition would $\overset{\frown}{AE} = \overset{\frown}{BE}$? What axiom makes this condition true in the drawing? Explain.

Proof:

Left to the pupil.

235. COROLLARY 1. *A diameter that bisects a chord (not a diameter) is perpendicular to the chord.*

236. COROLLARY 2. *The perpendicular bisector of a chord passes through the center of the circle.* (See § 203, *b*.)

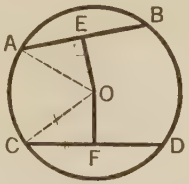
237. COROLLARY 3. *The perpendicular drawn to any chord from the center bisects the chord and the arcs subtended by it.*

Ex. If the center of a circle is joined to the mid-point of any chord and this line is extended in both directions to meet the circle, state the conclusions that follow.

Proposition V. Theorem

238. (*a*) *If in the same circle or in equal circles two chords are equal, they are equally distant from the center;* and

(*b*) Conversely, *if in the same circle or in equal circles two chords are equally distant from the center, they are equal.*



(*a*) **Given**: Circle O, chord AB = chord CD, $OE \perp AB$, $OF \perp CD$.

To prove: $OE = OF$.

(*b*) **Given**: Circle O, chords AB and CD, $OE \perp AB$, $OF \perp CD$, and $OE = OF$.

To prove: $AB = CD$.

Proof:

In both cases left to the pupil.

Discussion:

Consider the following possible positions of the two chords, make suitable drawings in each case, and prove:

1. The two chords parallel.
2. The two chords intersecting within the circle.
3. The two chords drawn from the same point on the circle.

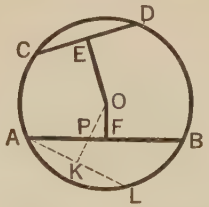
Ex. 1. In an inscribed equilateral triangle the three sides are equally distant from the center.

Ex. 2. All diameters of a circle are equal.

Ex. 3. If two boards are sawed from the same circular log equal distances from the center they will be of equal width. Explain.

Proposition VI. Theorem

239. *If in the same circle or in equal circles two chords are unequal, the greater chord is at the less distance from the center.*



Given: Circle O, chord $AB >$ chord CD, $OF \perp AB$, and $OE \perp CD$.

To prove: $OF < OE$.

Analysis:

1. This relation can be more easily proved if both chords are drawn from the same point. Draw chord AL = chord CD.
2. What line inequality must now be proved?
3. OF is at once shorter than what line in the drawing? Why?
4. Then how does OF compare with OK? with OE? Why?

Proof:

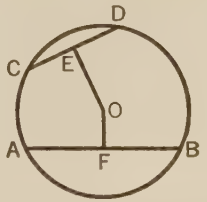
1. Minor arc $CD <$ minor arc AB.	1. Th. III, a
2. On minor arc AB take $\overset{\frown}{AL} = \overset{\frown}{CD}$. Draw AL.	2. ?
3. Draw $OK \perp AL$.	3. ?
4. Now chord AL = chord CD.	4. Th. II, b
5. $\therefore OK = OE$.	5. Th. V, a
6. Since AL and O lie on opp. sides of AB, OK must cross AB at some point, as P.	6. Cons.
7. Now $OF < OP$.	7. § 167
8. Also $OP < OK$.	8. Ax. 7
9. $\therefore OF < OK$.	9. Ax. 9
10. $\therefore OF < OE$.	10. Ax. 8

Discussion:

Suppose the greater chord passes through the center.

Proposition VII. Theorem

240. *If in the same circle or in equal circles two chords are unequally distant from the center, the chord at the less distance is the greater.*



Given: Circle O, chords AB and CD, $OF \perp AB$, $OE \perp CD$, and $OF < OE$.

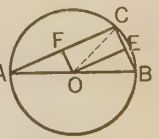
To prove: $AB > CD$.

Analysis:

1. What are the three possible relations that may exist between AB and CD?
2. Which do we wish to prove true? Hence which must be shown impossible?
3. Show that either of these relations, if true, would contradict the hypothesis.

Proof:

Left to the pupil.



Ex. 1. Given: Circle O, diameter AB, and chords AC and BC, BC being less than the radius.

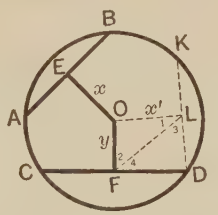
Prove: $OE > OF$.

* **Ex. 2.** A diameter of a circle is greater than any other chord of that circle. (Use Th. VII.)

Ex. 3. The distance from the center of a circle to a side of an inscribed equilateral hexagon is less than the distance from the center of the circle to a side of an inscribed equilateral octagon.

241. An Alternative Proof for Theorems VI and VII.

THEOREM VI



Given: Circle O, chords AB and CD, $OE \perp AB$, $OF \perp CD$, and $CD > AB$.
To prove: $y < x$.

Analysis:

1. $FD > LD$. Why?
2. $y < x'$ if $\angle 1 < \angle 2$. Why?
3. $\angle 1 < \angle 2$ if $\angle 3 > \angle 4$. Why?
4. But $\angle 3 > \angle 4$. Why?
5. $\therefore y < x'$, or $y < x$.

The pupil is to fill in the proof.

THEOREM VII

Given: Circle O, chords AB and CD, $OE \perp AB$, $OF \perp CD$, and $y < x$.
To prove: $CD > AB$.

Analysis:

1. $y < x'$. Why?
2. $FD > LD$ if $\angle 3 > \angle 4$. Why?
3. $\angle 3 > \angle 4$ if $\angle 1 < \angle 2$. Why?
4. But $\angle 1 < \angle 2$. Why?
5. $\therefore FD > LD$ and $CD > AB$.

This proof may be used instead of the one given in the text.

242. A polygon is *inscribed in a circle* when all its vertices lie on the circle. The circle is then *circumscribed about the polygon.*

Exercises — Group 46

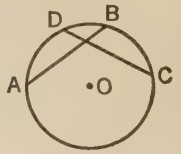
1. If an arc of a circle is doubled, is its central angle doubled? Is its chord doubled?

2. If a radius bisects an arc, does it also bisect its chord and its central angle?

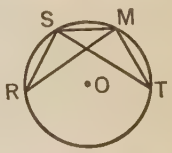
3. If the perpendiculars from the center to the sides of an inscribed polygon are equal, the polygon is equilateral.

4. If an equilateral polygon is inscribed in a circle, chords joining alternate vertices are equal.

5. Given: Chord AB = chord CD.
Prove: $\overset{\frown}{AD} = \overset{\frown}{CB}$.



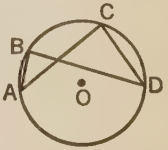
6. Points A, B, C, D are taken in order on a circle, AD being a diameter.
Prove: Chord BD > chord CD.



7. Given: $\overset{\frown}{RS} = \overset{\frown}{TM}$.
Prove: $\triangle RSM \cong \triangle TMS$.

8. If two diameters are perpendicular to each other, any chord parallel to one diameter is bisected by the other diameter.

9. Given: Circle O, chord BD > chord AC, and $\overset{\frown}{ABC}$ and $\overset{\frown}{BCD}$ minor arcs.
Prove: Chord CD > chord AB.



10. State and prove the converse of ex. 9.

* **11.** If a chord is drawn parallel to a diameter, the arcs intercepted between the chord and the diameter are equal.

SUG. Draw radii to the ends of the chord.

12. Given: Chord CA = chord CB, chord DA = chord DB.
Prove: CD is a diameter.

* **13.** If two intersecting chords make equal angles with the diameter through the point of intersection, the chords are equal.

Consider the following cases:

(a) When the two chords intersect on the circle.
(b) When the two chords intersect within the circle.
(c) When the two chords (extended) intersect without the circle.

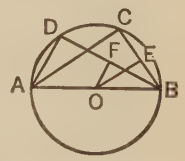
14. State and prove the converse of ex. 13; use the three cases.

15. If through any point in a diameter of a circle a chord is drawn perpendicular to the diameter, this chord is shorter than any other chord drawn through that point. (See § 240.)

16. If E and F are the mid-points of two parallel chords of a circle, show that the line through E and F passes through the center of the circle.

17. Given: Circle O, diameter AB, chords AC and BD, $OE \perp BC$, $OF \perp BD$, and $OE > OF$.

Prove: $AC > AD$.

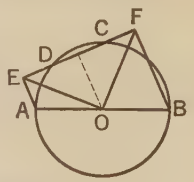


18. If two unequal chords intersect on a circle, the greater chord makes the less angle with the diameter through the point of intersection.

19. Given: Circle O, diameter AB, any chord DC, AE and BF each $\perp DC$ extended.

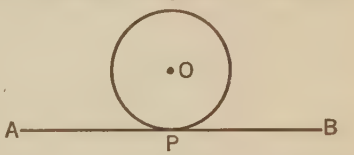
Prove: (*a*) $ED = FC$;

(*b*) $EO = FO$.



243. A *tangent* to a circle is a straight line which, however far it may be produced, has one and only one point in common with the circle. This point is called the *point of contact* or *point of tangency*.

Thus, in the figure, AB is a tangent to circle O and P is the point of contact.

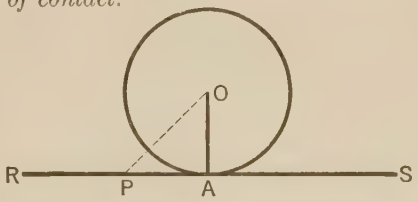


244. All points on a tangent to a circle, except the point of contact, are outside the circle.

No point on the circle, except the point of contact, could be on the tangent, by definition, and if any point inside the circle were on the tangent, the tangent would have at least two points in common with the circle (§ 223, 5), which would be impossible.

Proposition VIII. Theorem

245. *A tangent to a circle is perpendicular to the radius at the point of contact.*



Given: Circle O, line RS tangent to O at A, and radius OA.
To prove: $OA \perp RS$.

Analysis:

1. To be $\perp RS$, how must OA compare with any other line from O to RS, such as OP? (See § 167, *b*.)
2. Where is point P with reference to the circle? Why?
3. Hence compare OA and OP. Is $OA \perp RS$? Why?

Proof:

Left to the pupil.

246. Corollary 1. *A straight line perpendicular to a radius at its extremity on the circle is a tangent to the circle.*

Given: Circle O, radius OA, and line $RS \perp OA$ at A.
To prove: RS is a tangent to circle O.

Analysis:

1. If every point P on RS, excepting A, is outside the circle, RS is a tangent. (See § 243.)
2. Compare OP and OA. Hence where is P? (See § 223, 4.)

247. Corollary 2. *A perpendicular to a tangent at the point of contact passes through the center of the circle.*

See § 92.

248. Corollary 3. *The perpendicular drawn from the center of a circle to a tangent passes through the point of contact.*

See § 122.

249. A polygon is *circumscribed about a circle* when its sides are tangent to the circle. The circle is then *inscribed in the polygon.*

* **Ex. 1.** At a given point on a circle only one tangent can be drawn to the circle.

Ex. 2. Tangents to a circle at the ends of a diameter are parallel.

Ex. 3. Prove that a chord is parallel to the tangent drawn at the mid-point of its arc.

Ex. 4. If a polygon is circumscribed about a circle, each side is the same distance from the center.

Ex. 5. Two tangents to a circle meet in a point. Show that the center of the circle is on the bisector of the included angle.

Ex. 6. A straight line is tangent to circle O at point A. From any point P in this tangent, PB is drawn to point B on the circle so that $PA = PB$. Prove that PB is a tangent.

Sug. Show by $\cong$ $\triangle$ that $\angle OBP$ is a rt. $\angle$.

Ex. 7. Two unequal chords of circle O intersect at P within the circle. If E is the center of the shorter chord and F the center of the longer chord, prove: $EP < FP$.

Sug. Draw OE, OF, and EF.

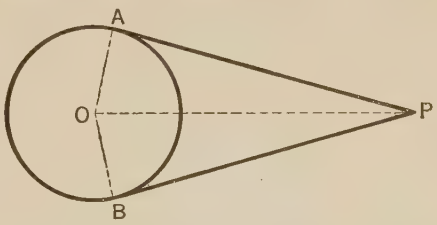
Ex. 8. In ex. 7, prove that the shorter chord makes the greater angle with the diameter through P.

Ex. 9. If two chords intersect within a circle and make unequal angles with the radius drawn through their point of intersection, they are unequal.

250. By the *length of a tangent* from a point to a circle is meant the length of the line from that point to the point of contact.

Proposition IX. Theorem

251. *Tangents drawn to a circle from an external point are equal.*



Given: Circle O and tangents PA and PB, drawn from the external point P.

To prove: $PA = PB$.

Analysis:

1. What is the usual method of proving lines equal?
2. What cons. is necessary if this method is to be used?

Proof:

Left to the pupil.

252. Corollary. *The tangents drawn to a circle from an external point make equal angles with the line joining that point with the center of the circle.*

Discussion:

1. State various ways in which OA and OB might be drawn and in each case show what necessary relation follows.
2. Show that $\overset{\frown}{AB}$ (between O and P) is always a minor arc.

253. The line joining the points of contact of two tangents drawn from an external point to a circle is called the *chord of contact* of the two tangents.

Ex. 1. Two tangents from an external point form equal angles with their chord of contact.

Ex. 2. Show that the line joining the center of a circle to the point of intersection of two tangents bisects: (*a*) the chord of con-

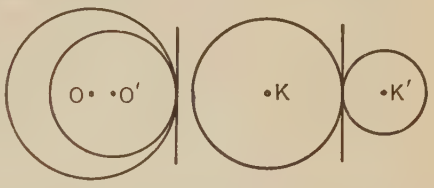
tact; (*b*) the angle formed by the radii drawn to the point of contact; (*c*) the arc intercepted by the tangents.

Two Circles

254. *Two circles are tangent* to each other if both are tangent to the same line at the same point.

255. Two circles are *tangent internally* or *externally*, according as one of the tangent circles lies wholly within or without the other.

Thus, in the drawings, circles O and O' are tangent internally and circles K and K' are tangent externally.



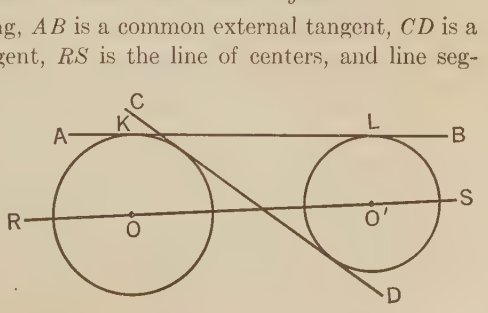
256. A line that touches two circles is called a *common internal tangent* if both circles lie on opposite sides of the line.

257. A line that touches two circles is called a *common external tangent* if both circles lie on the same side of the line.

258. The *length of a common tangent* is the length of the line between the points of contact.

259. The line determined by the centers of two circles is called the *line of centers* or the *center line*. The line segment joining the two centers is called the *center segment*.

Thus, in the drawing, AB is a common external tangent, CD is a common internal tangent, RS is the line of centers, and line segment OO' is the center segment. By the length of tangent AB is meant the length of the segment KL between the points of contact.



Exercises — Group 47

1. How many common tangents have two circles in each of the following positions? Make a suitable drawing in each case.

(*a*) If one circle lies wholly without the other.

(*b*) If the two circles are tangent externally; internally.

(*c*) If the two circles intersect.

(*d*) If the two circles are not tangent and one lies within the other.

2. Can two equal circles be tangent externally? internally?

3. Can the center segment of two circles be:

(*a*) Greater than the sum of the two radii?

(*b*) Equal to the sum of the two radii?

(*c*) Less than the sum of the two radii?

(*d*) Equal to the difference between the two radii?

(*e*) Less than the difference between the two radii?

Make a suitable drawing in each case and give reasons.

4. Show that the length of a common tangent of two circles is never greater than the center segment. Is it ever equal to the center segment? Explain.

5. The common internal tangents to two circles are equal.

6. The common external tangents to two circles are equal.

(*a*) When would they be parallel? why equal in this case?

(*b*) When not parallel, extend them to meet. (See § 251.)

7. Show that two common tangents to two circles make equal angles with their chords of contact.

8. Show that the line of centers is the perpendicular bisector of the chords of contact of two common tangents.

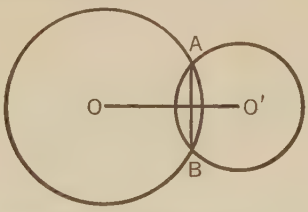
* **9.** Show that when two circles are tangent to the same line at the same point, tangents to the two circles from any point in this line are equal.

* **10.** If two circles are tangent externally, prove that the common internal tangent bisects each of the two external tangents and the segment included between them is equal to each of them.

260. The *common chord* of two intersecting circles is the chord joining the points of intersection.

Proposition X. Theorem

261. *When two circles intersect each other, the line of centers is the perpendicular bisector of the common chord.*



Given: Two intersecting circles O and O', OO' the center segment, and AB the common chord.

To prove: OO' is the perpendicular bisector of AB.

Analysis:

See § 204.

Proof:

Left to the pupil.

* **Ex. 1.** If two circles O and O' are tangent at point A, the line of centers passes through the point of contact.

Sug. Draw the common tangent, join A to O and O', and show that A, O, and O' are on one straight line.

Ex. 2. If three equal circles intersect in pairs, the three common chords pass through the same point.

Sug. Join the centers and use § 207.

262. *Concentric circles* are circles which have the same center.

Exercises — Group 48

1. Can two intersecting circles have the same center?

2. Can two equal circles be concentric and not coincide?

3. Radii of two circles drawn to the points of contact of a common tangent are parallel.

* **4.** In two concentric circles all chords of the one circle that are tangent to the other circle are equal.

5. If a line cuts two concentric circles, the segments of the line intercepted between the two circles are equal. (See § 234.)

6. Two circles O and O' are tangent and M is the mid-point of a common external tangent.

Prove: $\angle OMO'$ is a right angle.

Sug. Draw the common internal tangent.

* **7.** If two circles, tangent externally at P, touch a given line at points A and B, prove that $\angle BPA$ is a right angle. (See ex. 3, p. 113.)

8. Two circles are tangent to each other externally and each is tangent to a third circle internally. Show that the sum of the sides of the triangle formed by joining the three centers is equal to the diameter of the third circle.

9. If an equilateral pentagon is inscribed in a circle, all its diagonals are equal.

10. Any two parallels that cut a circle intercept equal arcs on the circle. (See ex. 11, p. 144.)

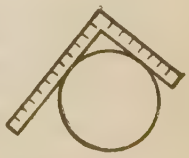
11. Circle O is inscribed in a triangle ABC and touches sides AB, BC, and CA at K, R, and L respectively.

(*a*) Mention three pairs of equal lines.

(*b*) Prove: $AB = AL + BR$ and $BC = BK + CL$.

12. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two opposite sides.

13. Show that if a circular block of wood is placed in the angle of a square and the two laid flat on a table, the edges of the square will touch the block at equal distances from the heel.



Could this be used to test whether the block is a true circle?

14. If the angle between two tangents drawn from a point to a circle is 60°, either tangent is equal to the chord of contact.

MEASUREMENTS OF ANGLES IN A CIRCLE

263. In § 41 it was shown how the unit for measuring angles is determined. When once the unit is determined, *any magnitude is measured* by finding how many times it contains this unit.

264. The *numerical measure* of any magnitude is the number which tells how many times this magnitude contains the given unit.

265. The *ratio* of two magnitudes of the same kind is the quotient or indicated quotient obtained by dividing the first magnitude by the second. To effect this division, it is therefore necessary first to obtain the numerical measure of each magnitude *in terms of the same unit.*

Thus, to find the ratio of 2′ to 1′ 6″, we express each magnitude in terms of a common unit and then find the ratio of these numerical measures.

$$\frac{2'}{1'\,6''} = \frac{24''}{18''} = \frac{4}{3}.$$

266. From this we may conclude: *Two magnitudes of the same kind have the same ratio as their numerical measures.*

Since a ratio is always an *abstract* number, it is possible to compare the ratio of two magnitudes of one kind and the ratio of two magnitudes of an entirely different kind.

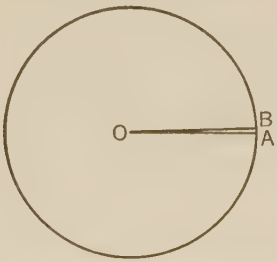
For example, the ratio of 2′ to 1′ 6″ is equal to the ratio of 1 lb. to 12 oz. since in each case the ratio is $\frac{4}{3}$.

267. Two magnitudes of the same kind do not always have a common unit of measure and hence their ratio cannot always be exactly expressed. Thus, the length of a circle and its diameter are two magnitudes which have no common unit of measure. It is therefore impossible to express their ratio exactly. In fact this ratio equals the never-ending decimal 3.14159 . . . (See §§ 448, 477.)

268. Two magnitudes of the same kind which have a common unit of measure are said to be *commensurable.*

269. Two magnitudes of the same kind which have no common unit of measure are said to be *incommensurable.*

270. The unit for measuring angles is the *angle degree* (1°). If this unit angle is constructed at the center of a circle, the arc which it intercepts on the circle gives the unit for measuring arcs of a circle, called the *arc degree.*



Thus, in the drawing, if the angular magnitude at O is divided into 360 equal parts and $\angle AOB$ is one of these equal angles, then $\angle AOB$ is the unit for measuring angles and equals 1 angle degree (1°), while the corresponding arc AB is the unit for measuring arcs of a circle and equals 1 arc degree.

271. It is evident that for every angle degree constructed at the center there corresponds an arc degree on the circle, and hence every central angle contains the same number of angle degrees as its arc contains arc degrees. This is often stated:

272. *A central angle is measured by its intercepted arc.*

The truth of this statement will be assumed.

This statement means that the same number expresses the measure of a central angle and of its intercepted arc; but it does not mean that an angle is the same thing as an arc. It states that their *numerical measures* are the same, just as the *number* of people at an entertainment may equal the *number* of tickets taken in without implying that a person and a ticket are the same thing.

273. Corollary. *In the same circle or in equal circles two central angles have the same ratio as their intercepted arcs.*

Since the numerical measure of each angle is equal respectively to the numerical measure of its intercepted arc, the ratio of the angles equals the ratio of the arcs.

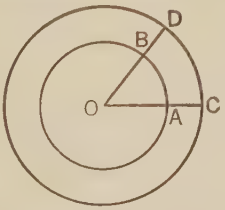
In the proof of this corollary it is tacitly assumed that the ratio between the two angles *can be exactly expressed;* that is, that the angles are commensurable. The statement of the corollary is equally true when the angles are incommensurable. The truth of the statement in this case, however, will here be assumed.

274. The statement that a central angle is measured by its intercepted arc is in reality a *form of equality,* as will readily appear if it is put in the following form:

Numerical measure of angle = numerical measure of its arc.

Hence all the usual laws of operations on equals may be applied to this statement. For example, half a central angle is measured by half its intercepted arc, by Ax. 5, etc.

275. The number of arc degrees in an arc must not be confused with the number of feet or inches in the arc; that is, with its length. Two arcs in different circles may have the same number of arc degrees, but one arc may be much longer than the other.



Thus, in the drawing, $\overset{\frown}{CD}$ and $\overset{\frown}{AB}$ have the same number of arc degrees (in each case equal to the number of angle degrees in $\angle O$), but $\overset{\frown}{CD}$ is manifestly longer than $\overset{\frown}{AB}$. In the same circle or in equal circles, however, arcs having the same number of arc degrees are also equal in length.

276. A *secant* is an unlimited straight line that cuts the circle in two points. (See Fig. 1 below.)

Unless otherwise stated, the *length of a secant* from an external point to a circle means the length of the line from the external point to the second point of intersection.

It is to be noted that a tangent is the *final (limiting) position of a secant* when its two points of intersection coincide.

277. An *inscribed angle* is an angle whose vertex is on the circle and whose sides are chords. (See Fig. 1 below.)

278. A *segment* of a circle is the figure bounded by an arc of a circle and its chord. (See Fig. 2 below.)

If the arc is a major arc, the segment is called a *major segment.* If the arc is a minor arc, the segment is called a *minor segment.*

279. An angle is said to be *inscribed in a segment* or *inscribed in an arc* if its vertex is on the arc and its sides terminate in the ends of the arc. (See Fig. 2 below.)

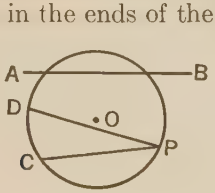


Fig. 1

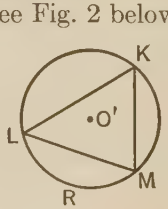


Fig. 2

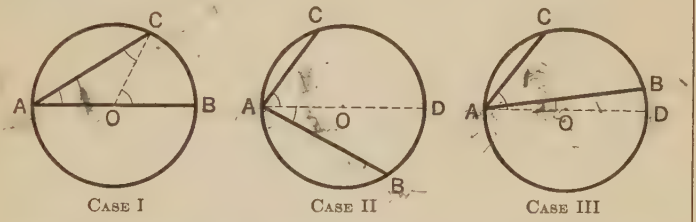
Thus, in Fig. 1, AB is a secant and $\angle CPD$ is an inscribed angle; also, in Fig. 2, LRM is a segment (in this case a minor segment) and $\angle LKM$ is said to be inscribed in segment LKM.

Ex. 1. Using § 272, find the angle at the center of a circle subtended by one side (*a*) of an inscribed equilateral triangle; (*b*) of an inscribed square; (*c*) of an inscribed equilateral hexagon.

Ex. 2. Central angles of the sides of an inscribed triangle are in the ratio 2 : 3 : 5. Find the number of degrees in the intercepted arcs.

Proposition XI. Theorem

280. *An angle inscribed in a circle is measured by one half its intercepted arc.*



Case I Case II Case III

Given: Circle O and inscribed angle $\angle BAC$ (three cases).
To prove: $\angle BAC$ is measured by $\frac{1}{2}\overset{\frown}{BC}$.

Three cases are recognized, according as the center of the circle is in one arm of the angle, within the angle, or outside the angle.

Case I. Center O in AB, one arm of the angle.

Analysis:

1. What class of $\measuredangle$ in circles has been considered whose measurement is known?
2. How can such an $\angle$ be formed in the drawing?
3. In terms of the drawing how is this $\angle$ measured? Why?
4. Express the required $\angle$ in terms of this $\angle$.
5. Hence how is the required $\angle$ measured? (See § 274.)

Proof:

For all cases left to the pupil.

Case II. Center of the circle within the angle.

Case III. Center of the circle outside the angle.

Analysis:

1. Show how each case can be made to depend on Case I.
2. What axiom should then be used?

281. COROLLARY 1. *All angles inscribed in the same segment or in equal segments are equal.*

Show that all such angles intercept the same arc or equal arcs.

282. COROLLARY 2. *An angle inscribed in a semicircle is a right angle.*

How many arc degrees are there in the intercepted arc?

283. COROLLARY 3. *An angle inscribed in a segment whose arc is greater than a semicircle is an acute angle. An angle inscribed in a segment whose arc is less than a semicircle is an obtuse angle.*

In each case note the size of the arc intercepted by the angle.

Exercises — Group 49

1. In Case I, p. 157, if $\angle A = 40°$, how many arc degrees are there in $\overset{\frown}{AC}$? If $\overset{\frown}{BC} = 2\ \overset{\frown}{AC}$, find $\angle A$.

2. In Case II, p. 157, if $\overset{\frown}{AC} = 84°$ and $\overset{\frown}{DC} = 3\ \overset{\frown}{BD}$, find $\angle BAC$.

3. In Case III, p. 157, if C is the center of $\overset{\frown}{AB}$ and $\overset{\frown}{DB} = \frac{1}{4}\ \overset{\frown}{BC}$, find $\angle BAC$.

4. Two angles of an inscribed triangle are 50° and 80°. Find the number of degrees in the arcs subtended by the three sides.

5. Three consecutive sides of an inscribed quadrilateral subtend arcs of 64°, 100°, and 48°. Find each angle of the quadrilateral and the angle between the diagonals.

*** 6.** A circle is circumscribed about triangle ABC and M is the mid-point of arc AB.

Prove: $\angle ABM = \frac{1}{2}\ \angle C$.

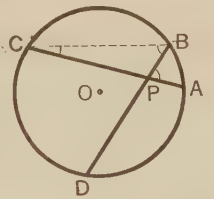
7. The quadrilateral formed by joining in order the ends of any two diameters is a rectangle.

*** 8.** A circle is described upon side AB of triangle ABC as a diameter and cuts side AC, produced if necessary, in point D. Prove that BD is perpendicular to AC.

9. An inscribed angle ABC is 30°. Show that chord AC is equal to a radius.

Proposition XII. Theorem

284. *An angle formed by two chords intersecting within a circle is measured by one half the sum of the arc intercepted between its sides and the arc intercepted between the sides of its vertical angle.*



Given: Circle O and chords AC and BD intersecting in point P within the circle.

To prove: $\angle APB$ is measured by $\frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{CD})$.

Analysis:

1. Inscribed $\measuredangle$ will probably be used. What cons. would form such $\measuredangle$ intercepting the two arcs in question?
2. How is each of these $\measuredangle$ measured? Why?
3. Express the required $\angle$ in terms of these $\measuredangle$.

Proof:

Left to the pupil.

Discussion:

1. What other line might have been drawn instead of BC?
2. What cons. would show that $\angle BPC$ is measured by $\frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{DA})$?
3. Show how § 280 might be considered a special case of § 284.

Ex. 1. If $\overset{\frown}{AB} = 50°$, $\overset{\frown}{BC} = 140°$, and $\overset{\frown}{CD} = 150°$, find $\angle DPA$.

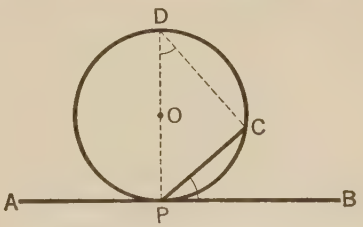
Ex. 2. If $\overset{\frown}{CD} = 80°$, $\overset{\frown}{DA} = 70°$, and $\overset{\frown}{BC} = 3\,\overset{\frown}{AB}$, find $\angle BPC$.

Ex. 3. Check ex. 5, p. 158, by § 284.

Ex. 4. The sides AB, BC, and CA of the inscribed triangle ABC subtend arcs of 172°, 64°, and 124° respectively. Find the angle formed at the intersection of AB and the bisector of angle C.

Proposition XIII. Theorem

285. *An angle formed by a tangent and a chord drawn from the point of contact is measured by one half the intercepted arc.*



Given: AB tangent to circle O at point P and PC any chord through P.

To prove: $\angle BPC$ is measured by $\frac{1}{2}\overset{\frown}{PC}$.

Analysis:

1. Form an inscribed angle D intercepting this arc PC.
2. How is $\angle D$ constructed? How is it measured? Why?
3. Compare $\angle D$ and $\angle BPC$. (See § 93.)

Proof:

Left to the pupil.

Discussion:

1. What is the measure (in arc degrees) of $\angle BPD$?
2. Prove § 285, starting with $\angle BPC = \angle BPD - \angle CPD$.
3. Prove that $\angle CPA$ is measured by $\frac{1}{2}\overset{\frown}{CDP}$. (See Ax. 6.)
4. Suppose that PC is a diameter.

Ex. 1. If $\overset{\frown}{PC}$ equals 78°, how many degrees are there in $\angle BPC$? If $\angle BPC = 54°$, how many degrees are there in arc PC?

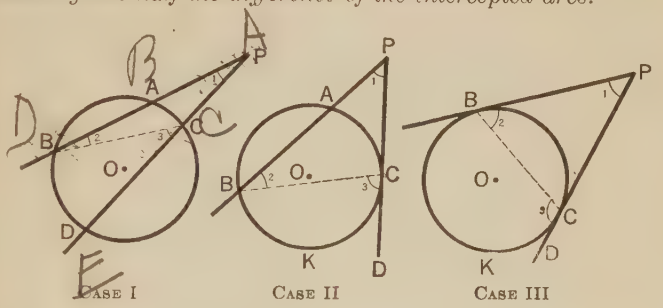
Ex. 2. Tangents to a circle at the ends of a chord make equal angles with the chord.

Ex. 3. A tangent at point P on a circle is parallel to chord AB. Prove that $\triangle PAB$ is isosceles.

Ex. 4. The mid-point of the arc of any chord is equally distant from the chord and the tangent drawn at either end of the chord.

Proposition XIV. Theorem

286. *An angle formed by two secants, a secant and a tangent, or two tangents intersecting outside a circle, is measured by one half the difference of the intercepted arcs.*



Case I Case II Case III

Case I. Two Secants.

Given: Circle O and PAB and PCD two secants intersecting at P, a point outside the circle.

To prove: $\angle P$ is measured by $\frac{1}{2}(\overset{\frown}{BD} - \overset{\frown}{CA})$.

Analysis:

1. What cons. would form inscribed $\measuredangle$ intercepting the arcs in question?
2. How is each of these $\measuredangle$ measured? Why?
3. Express the required $\angle$ in term of these $\measuredangle$.

Proof:

For all cases left to the pupil.

Case II. A Secant and a Tangent.

Case III. Two Tangents.

In Cases II and III use an outline similar to the one for Case I.

Exercises — Group 50

1. If $\overset{\frown}{BD}$ (Case I, p. 161) is 150° and $\overset{\frown}{CA}$ is 36°, find the number of degrees in $\angle P$.

2. If $\overset{\frown}{CA}$ (Case I, p. 161) is 48° and $\angle P = 20°$, find the number of degrees in $\overset{\frown}{BD}$.

3. If $\overset{\frown}{BKC}$ (Case III, p. 161) is 3 times $\overset{\frown}{CB}$, find the number of degrees in $\angle P$. What is the sum of these two arcs?

4. If $\angle P$ (Case III, p. 161) is 70°, find $\overset{\frown}{BKC}$ and $\overset{\frown}{CB}$.

5. If $\angle P$ (Case II, p. 161) is 42° and $\overset{\frown}{BKC}$ and $\overset{\frown}{CA}$ are supplementary, find the number of degrees in each arc.

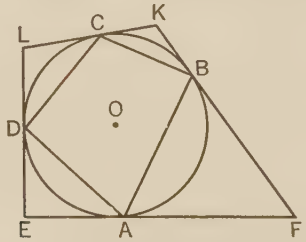
6. If $\angle P$ (Case II, p. 161) is 50° and $\overset{\frown}{AB}$ is 40°, find the number of degrees in $\overset{\frown}{AC}$ and also in $\overset{\frown}{BKC}$.

7. If $\overset{\frown}{BKC}$ (Case III, p. 161) is 210°, find $\angle P$.

8. If two chords intersect at right angles, the sum of either opposite pair of arcs equals a semicircle.

9. If a line is tangent to a circle and two chords are drawn from the point of contact making equal angles with the tangent, the chords are equal.

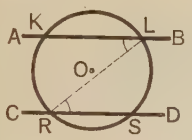
10. $ABCD$ is an inscribed quadrilateral, $EFKL$ is a circumscribed quadrilateral, $\overset{\frown}{AB} = 130°$, $\overset{\frown}{BC} = 64°$, $\overset{\frown}{CD} = 80°$. Find: (1) $\overset{\frown}{DA}$; (2) the three angles at A, at B, at C, and at D; (3) $\angle E$,



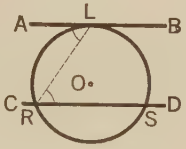
$\angle F$, $\angle K$, $\angle L$; (4) the angles formed by the intersection of the chords AC and BD; (5) the angle between AB and DC (produced to meet); (6) the angle between EF and CB (produced to meet).

Proposition XV. Theorem

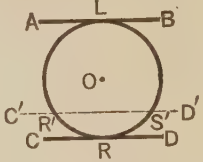
287. *Arcs of a circle intercepted by two parallel lines are equal.*



Case I



Case II



Case III

Case I. Two Parallel Secants.

Given: Circle O and two $\parallel$ secants AB and CD intercepting the arcs KR and LS.

To prove: $\overset{\frown}{KR} = \overset{\frown}{LS}$.

Analysis:

1. What cons. would make these arcs the arcs of two inscribed $\measuredangle$?
2. How do these $\measuredangle$ compare? Why? How are they measured?
3. Hence how do the arcs compare?

Proof:

For all cases left to the pupil.

Case II. A Secant and a Tangent Parallel.

Use an outline similar to that of Case I.

Case III. Two Parallel Tangents.

Use Case II and Ax. 2.

Exercises — Group 51

*** 1.** Two parallel chords intercept equal arcs on a circle.

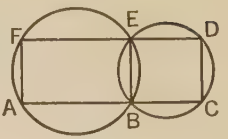
2. Is the converse of ex. 1 true? Prove your statement.

*** 3.** A trapezoid inscribed in a circle is isosceles.

*** 4.** A parallelogram inscribed in a circle is a rectangle.

*** 5.** If two opposite sides of an inscribed quadrilateral are equal and parallel, the figure is a rectangle.

6. Given: $FED \parallel ABC$.
Prove: $FA = EB = DC$.



288. The preceding theorems show various ways of proving lines, angles, or arcs equal or unequal when the parts involved are found *in the same circle or in equal circles.* Under these conditions:

I. *Two lines are equal if they are:*
1. Chords of equal central angles or of equal arcs.
2. Chords the same distance from the center.
3. Tangents from the same external point.

II. *Two angles are equal if they are:*
1. Central angles whose chords or arcs are equal.
2. Angles with vertices on the circle intercepting the same or equal arcs.

III. *Two arcs are equal if they are:*
1. Arcs of equal central angles or of equal chords.
2. Arcs intercepted by parallel lines.
3. Arcs intercepted by equal angles whose vertices are on the circle.

IV. *Two lines are unequal if they are:*
1. Chords of unequal central angles or of unequal arcs.
2. Chords unequally distant from the center.

V. *Two angles are unequal if they are:*
1. Central angles whose chords or arcs are unequal.
2. Angles with vertices on the circle intercepting unequal arcs.

VI. *Two arcs are unequal if they are:*
Arcs of unequal central angles or of unequal chords.

Exercises — Group 52

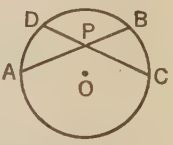
Lines Equal

1. If from either end of a diameter chords are drawn making equal angles with the diameter, the chords are equal.

2. If two chords are equal and their end points are joined, two of these lines will be parallel and the other two equal. (Take the given chords in various positions.)

3. State and prove the converse of ex. 2.

4. Given: Chord AB = chord CD.
Prove: $AP = CP$; $BP = DP$.



5. If two secants to a circle from an external point are equal, the intercepted chords are equal.

6. What is the converse of ex. 5? Is it true? Explain.

7. If BD, the bisector of an inscribed angle ABC, is produced to meet the circle in D and through this point a chord AB is drawn parallel to one side of the angle, the chord will equal the other side of the angle.

8. AB and AC are chords of circle O on opposite sides of center O. D and E are the mid-points of arcs AB and AC respectively. Show that DE cuts off equal distances on the chords from A.

Exercises — Group 53

Angles Equal

1. The angle between a tangent and a chord drawn from the point of contact is equal to one half the angle between the radii to the ends of the chord.

2. Given circle O, diameter AD, chord AB, and C the mid-point of arc BD. Show that OC is parallel to AB.

3. In the same circle or in equal circles two inscribed angles are equal if their sides are equal respectively.

4. Two secants PAB and PCD cut the circle O in points A, B and C, D respectively. Prove that the angles of triangle PBC are equal respectively to the angles of triangle PAD.

5. If two chords do not intersect within the circle and the midpoint of the minor arc of one chord is joined to the ends of the other chord, the two triangles formed have their angles equal respectively.

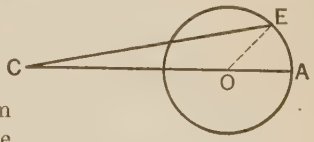
6. Two tangents from P touch circle O at A and B.
Prove: $\angle APO = \angle BAO$.

Exercises — Group 54

Lines and Angles Unequal

1. Of all secants from an external point to a circle, the longest passes through the center of the circle.

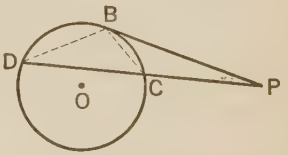
Prove: $CA > CE$.



2. The tangent to a circle from an external point is longer than the external segment of any secant drawn from the same point to the circle.

Prove: $PB > PC$.

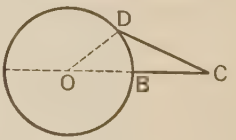
Sug. See § 166.



* **3.** The shortest line that can be drawn from a point outside a circle to the circle is the line which, if extended, passes through the center of the circle.

Prove: $CB < CD$.

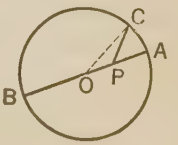
Sug. $OC < (OD + DC)$.



* **4.** The shortest line that can be drawn from a point within a circle to the circle is the shorter segment of the diameter through that point.

Prove: $PA < PC$.

Sug. $(OC - OP) < PC$. Why?



5. The longest line that can be drawn from a point within a circle to the circle is the longer segment of the diameter through that point.

In the figure of ex. 4, prove: $PB > PC$.

6. PDA and PCB are two secants from point P cutting circle O in points D, A and C, B respectively. Show that any inscribed angle of circle O intercepting arc AB is greater than angle P.

Exercises — Group 55

Inscribed and Circumscribed Polygons

1. If two sides of an inscribed quadrilateral are parallel, the other two sides are equal.

2. Is the converse of ex. 1 true? Prove your statement.

*** 3.** The opposite angles of an inscribed quadrilateral are supplementary.

4. An exterior angle formed at any vertex of an inscribed quadrilateral is equal to the opposite angle of the quadrilateral.

5. If a triangle is inscribed in a circle and a tangent is drawn at any vertex, the angles formed between the tangent and the sides will equal the other two angles of the triangle.

*** 6.** The tangent at the vertex of an inscribed isosceles triangle makes equal angles with the legs and is parallel to the base.

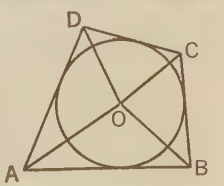
7. A rectangle circumscribed about a circle is a square.

8. The sum of two opposite sides of a circumscribed quadrilateral equals the sum of the other two opposite sides.

9. If an isosceles triangle is circumscribed about a circle, the lines joining the points of tangency form an inscribed isosceles triangle.

10. Either pair of opposite sides of a circumscribed quadrilateral subtend at the center of the circle two supplementary angles.

Prove: $\angle AOB + \angle COD = 1$ st. $\angle$.



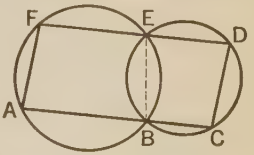
* **11.** The radius of the circle inscribed in an equilateral triangle is one third the altitude of the triangle.

* **12.** The radius of the circle circumscribed about an equilateral triangle is two thirds the altitude of the triangle.

Exercises — Group 56

Two Circles

* **1.** If two circles are concentric, chords of the larger circle tangent to the smaller circle are bisected at the point of contact.



2. Given: $FED \parallel ABC$.
Prove: $ACDF$ is a parallelogram.

3. Two circles intersect in A and B. From point P in one circle PAC and PBD are drawn terminating in the other circle. Prove that CD is parallel to the tangent at P.

4. Two equal circles intersect in A and B and through A any line CAD is drawn, meeting the circles in C and D.
Prove: $BC = BD$.

* **5.** Two circles are tangent internally at P, the diameter of the smaller circle being equal to the radius of the larger circle. Prove that any chord of the larger circle, drawn from P, is bisected by the smaller circle.

* **6.** Two circles are tangent internally or externally at P and through P two lines are drawn terminating in the two circles. Prove that the chords joining the ends of these lines are parallel.
Sug. Draw the tangent at P.

* **7.** A common external tangent touches two circles at P and P'. The line of centers cuts one circle in A and B, and the other circle in C and D respectively.
Prove: $AP \parallel CP'$.
Sug. Draw radii to P and P'.

8. PC is the common chord of two intersecting circles. If PA is tangent to one of the circles and PB is tangent to the other circle, prove that the triangles PAC and PBC are mutually equiangular.

Exercises — Group 57

MISCELLANEOUS

1. Prove Th. XI, Case I, p. 157, by drawing a diameter parallel to AC.

2. Prove Th. XII, p. 159, by drawing a chord through A parallel to DB.

3. Prove Th. XIII, p. 160, by drawing a chord through C parallel to AB.

4. Prove Th. XIV, p. 161, by drawing a chord through C parallel to PB.

* **5.** A circle described on one of the legs of an isosceles triangle as a diameter bisects the base.

* **6.** A circle described on the hypotenuse of a right triangle as a diameter passes through the vertex of the right angle.

7. A circle can always be circumscribed about any rectangle.

* **8.** A circle described on one side of a triangle as a diameter passes through the feet of the altitudes on the other two sides.

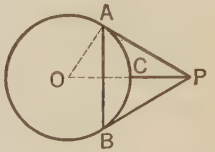
9. If BD and AE are two altitudes of triangle ABC, show that $\angle ABD = \angle AED$.

10. AB is the hypotenuse of right triangle ABC. O is the intersection of the diagonals of the square drawn on AB. Show that OC bisects angle C.

* **11.** Given: Tangents PA and PB and PC = a radius.

Prove: $\triangle PAB$ is equilateral.

SUG. See ex. 8, p. 123.



12. The sum of the legs of a right triangle equals the sum of the hypotenuse and the diameter of the inscribed circle.

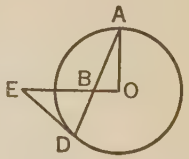
13. PAB and PD are a secant and a tangent respectively from point P to circle O, and DC is a chord parallel to PAB. Show that triangles PAD and DBC have their angles equal respectively.

14. Circle O inscribed in triangle ABC touches AB, BC, and CA at P, Q, and R respectively.

Prove: $\angle PQR + \frac{1}{2}\angle A = 90°$.

Sug. Join O to A, P, and R.

15. O is the center of a circle, OA any radius, and B any point on a radius perpendicular to OA. Line AB produced meets the circle at D and the tangent at D meets OB produced at E.

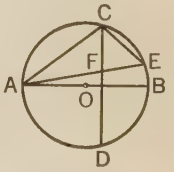


Prove: $\triangle EDB$ is isosceles.

16. From the same point on a circle two and only two equal chords can be drawn.

Sug. See § 238, *b* and § 240.

* **17.** Given: Diameter AB, $CD \perp AB$, E any point in arc CB, and F the intersection of AE and CD.



Prove: $\angle ACE = \angle DFE$.

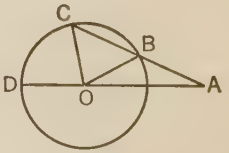
18. Tangents drawn at the vertices of an inscribed parallelogram form a rhombus.

* **19.** A parallelogram (not a rectangle) circumscribed about a circle is a rhombus.

20. Given: $OB = AB$.

Prove: $\angle COD = 3\ \angle A$.

Note. The fact here stated is used in the construction of certain mechanical devices for the trisection of any angle. It has been proved that, in general, an angle cannot be trisected by geometric methods.



21. Two radii at right angles to each other are extended to meet a tangent and from these points of intersection tangents are drawn to the circle. Prove that these tangents are parallel.

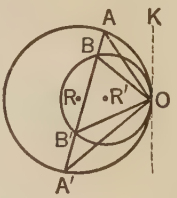
* **22.** Two tangents are drawn to a circle O at the ends of a diameter and intercept a segment PQ on a third tangent. Prove that $\angle POQ$ is a right angle.

23. Two circles O and O' intersect in P and Q, and diameters PA and PB are drawn. Prove that AQB is a straight line.

24. Two circles are tangent externally at P, and AB is a common external tangent. If the diameter from A meets the circle in C, show that CPB is a straight line.

SUG. Draw AP.

25. Two circles are tangent internally at O. A line cuts one circle in A and A' and the other circle in B and B'.



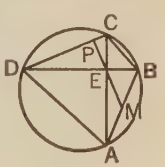
Prove: $\angle AOB = \angle A'OB'$.

SUG. Draw the common tangent. Then $\angle AOB = \angle KOB - \angle KOA$ and $\angle A'OB' = \angle BB'O - \angle B'A'O$, etc.

26. If two circles are tangent internally at P and a chord AB of the larger circle is tangent to the smaller circle at M, prove that line PM bisects $\angle APB$.

SUG. See ex. 25.

27. If the diagonals of an inscribed quadrilateral intersect at right angles, a straight line through their intersection that bisects one side is perpendicular to the opposite side.



Given: $AC \perp BD$, PEM bisecting AB.

Prove: $MP \perp DC$.

SUG. Show: $\angle ECP = \angle BEM$ and $\angle CEP = \angle MEA$ and hence $\angle EPC$ is a right angle.

28. ABC is an equilateral triangle inscribed in a circle and P is any point in arc AC.

Prove: $AP + PC = BP$.

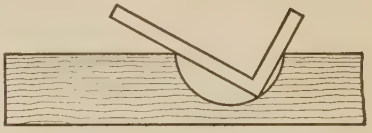
SUG. On PB take $PD = PA$ and join A to D.

***29.** Show that the two common external tangents and the two common internal tangents to two unequal circles intersect on the line of centers.

Exercises — Group 58

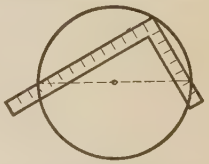
APPLICATIONS

1. Show how to determine with a carpenter's square whether a cut in a board is semicircular.

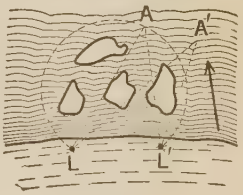


2. Show how to determine the radius of a small circular plate by use of an ordinary pair of slide calipers.

3. Show how to find the center and the radius of a circular piece of board with a carpenter's square.

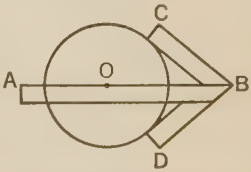


4. L and L' are lighthouses located on a shore near which are some dangerous rocks all situated within a circle passing through L and L'. Show that a ship is safe from the rocks so long as it keeps in a course in which $\angle A' < \angle A$.

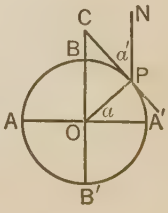


The angle A subtended in the circle is known to sailors as the *horizontal danger angle*.

5. The drawing shows an instrument for locating the center of a circular disk. Here CB and DB are equal and AB bisects the angle between BC and BD. When BC and BD touch a circular disk, AB will pass through its center. Why?



6. In the drawing AA' represents the equator, BB' the axis of the earth, P any place on the earth's surface, PC a plane tangent to the earth at P (the horizon), and PN the line of observation to the polar star. (For all practical purposes it may be assumed that $PN \parallel B'B$.) Then $\angle a$ is the latitude of P and $\angle a'$ the latitude of the polar star.



Show that the latitude of a place is equal to the latitude of the polar star; that is, prove $\angle a = \angle a'$.

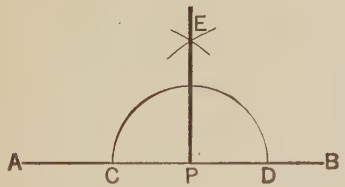
FUNDAMENTAL PROBLEMS OF CONSTRUCTION

289. The *solution of a construction problem* consists of four steps: (1) a statement of what is *given;* (2) a statement of what is *required* or what is to be done; (3) the *construction,* or explanation of how it is done; (4) the *proof,* showing that the construction is correct. When the problem permits, there is a fifth step — the *discussion* — wherein are considered the conditions under which the problem would have no solution or more than one solution.

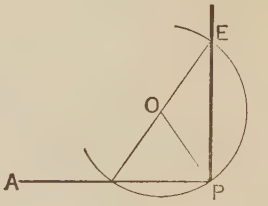
Articles 71 to 79 should now be carefully reviewed and the correctness of each construction established.

Proposition XVI. Problem

290. *To draw a perpendicular to a given line at a point in the line.*



Case I



Case II

Given: Line AP.

Required: To construct a perpendicular to AP at P.

Case I. The given point not at the end of the line.

Construction and Proof:

Left to the pupil.

Case II. The given point at the end of the line.

Construction:

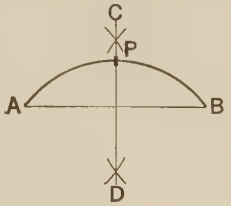
Use § 282.

Proof:

Left to the pupil.

Proposition XVII. Problem

291. *To bisect a given arc of a circle.*



Given: AB an arc of a circle.
Required: To bisect arc AB.

Construction:

See § 234.

Proof:

Left to the pupil.

* **Ex. 1.** Show how to construct each of the following angles without the use of the protractor:

(*a*) 60°	(*d*) 45°	(*g*) 75°
(*b*) 30°	(*e*) 22° 30′	(*h*) 120°
(*c*) 15°	(*f*) 135°	(*i*) 150°

Ex. 2. Show how to trisect a right angle.

Ex. 3. Show how a draftsman with his T-square, 30°–60° and 45° triangles, can construct angles of 15°, 75°, and 105°.

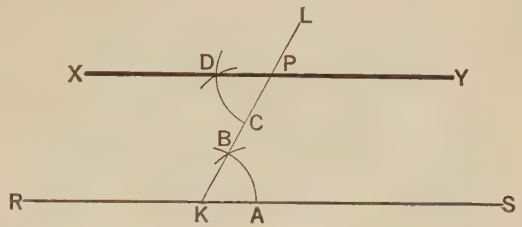
Ex. 4. Given an arc of a circle, required to find the center of the circle.

Sug. See § 236.

Ex. 5. S is a fragment of a circular wheel. Show how to determine the radius of the original wheel.

Proposition XVIII. Problem

292. *Through a given point to construct a line parallel to a given line.*



Given: Line RS and P any point outside RS.

Required: To construct a line through $P \parallel RS$.

Analysis:

1. What condition in § 137 makes the lines $\parallel$?
2. In this problem what auxiliary line must be drawn through P? Explain.
3. Then how shall the line be constructed through $P \parallel RS$? (See § 74.)

Construction and Proof:

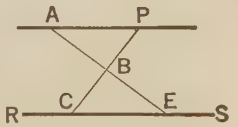
Left to the pupil.

Discussion:

1. Where would the angle be constructed if the problem were done by § 138?
2. Show how this problem could be done by § 140.

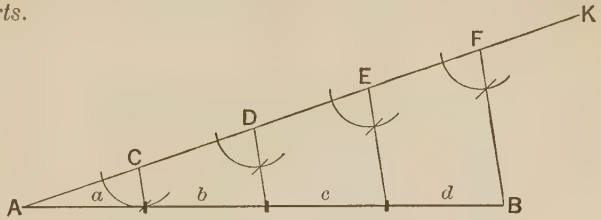
Ex. Using the following method, draw through a given point P a line parallel to a given line RS:

Choose any point C in RS, and find B halfway between P and C. Assume a point E in RS and make $BA = EB$. Then $PA \parallel RS$. Prove.



Proposition XIX. Problem

293. *To divide a given line into any number of equal parts.*



Given: Line segment AB.

Required: To divide AB into n equal parts. (In the diagram $n = 4$.)

The plan is to use § 191, drawing AK for the second transversal, marking off equal segments on AK, and then constructing the series of parallels.

Construction:

1. Draw AK, making any $\angle$ with AB.	1. ?
2. On AK, starting at A and using any length, mark off n equal segments in order, giving points C, D, E, F.	2. ?
3. Join the last point F with B.	3. ?
4. Through E, D, C draw $\parallel$s to FB, dividing AB into segments a, b, c, d.	4. ?

Proof:

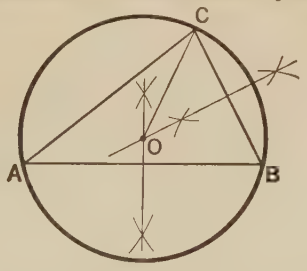
$a = b = c = d$.	§ 191

Discussion:

1. Does $a = AC$?
2. Must AB and AK intersect? Explain.
3. Why is BF drawn first?
4. Could this method be used to bisect a line?
5. Show that only the $\parallel$s through F and E need be drawn.
6. Show how with ruled paper a line may be divided into any number of equal parts.

Proposition XX. Problem

294. *To circumscribe a circle about a given triangle.*



Given: $\triangle ABC$.

Required: To circumscribe a circle about $\triangle ABC$.

Construction and Proof:

Left to the pupil. (See §§ 212 and 207.)

Discussion:

1. Where is the center when the $\triangle$ is acute? right? obtuse?
2. How many circles can be circumscribed about a given $\triangle$?
3. Can a circle always be circumscribed about a given $\triangle$?

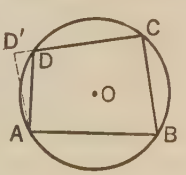
295. Corollary. *Through three points not in a straight line, one circle and only one can be passed.*

These points can be used as the vertices of a triangle. Why?

* **Ex. 1.** Two circles cannot intersect in more than two points.

Ex. 2. If two circles intersect in one point, they must intersect in two points.

* **Ex. 3.** If the opposite angles of a quadrilateral are supplementary, a circle can be circumscribed about it.



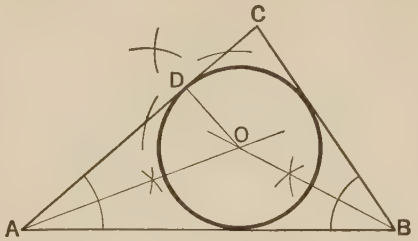
Sug. 1. Through how many vertices of the quad. can a circle be passed at once? Why?

2. Suppose the remaining vertex D' fell as in the drawing. Compare $\angle D'$ and $\angle B$; $\angle D$ and $\angle B$.

3. Hence show that D' and D must coincide.

Proposition XXI. Problem

296. *To inscribe a circle in a given triangle.*



Given: $\triangle ABC$.

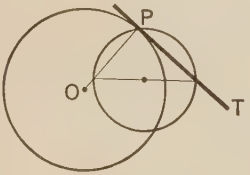
Required: To inscribe a circle in $\triangle ABC$.

Construction and Proof:

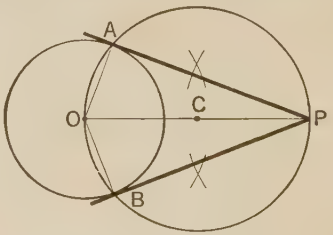
Left to the pupil. (See §§ 214 and 209.)

Proposition XXII. Problem

297. *Through a given point to construct a tangent to a given circle.*



Case I



Case II

Given: Circle O and point P.

Required: To construct a tangent to circle O through P.

Case I. The given point P on the circle.

Construction and Proof:

Left to the pupil. (See § 246 and § 290, Case II.)

CASE II. The given point P outside the circle.

Analysis:

1. The drawing when made will appear as in § 251.
2. What must be the size of $\angle A$? of $\angle B$? Why?
3. Locate points A and B by use of § 282.
4. What would be the center and the radius of the circle?

Construction and Proof:

Left to the pupil.

Discussion:

1. Will there always be two tangents when the point is outside the circle?
2. Could point C fall on the circle? within the circle? Explain.
3. Is $\angle P$ always acute? Is $\angle AOB$ always obtuse?
4. Could $\angle AOB$ ever be a right angle? Explain.

Ex. 1. In § 296, must the center of the circle always lie within the triangle? Explain.

Ex. 2. How many circles can be inscribed in a given triangle?

Ex. 3. A circle can be constructed that will touch each of three given lines if they are not all parallel. Prove.

Ex. 4. Show that there are three circles each of which touches the sides of any triangle externally.

Ex. 5. Show that in an equilateral triangle the inscribed and circumscribed circles are concentric.

Ex. 6. Show that in an acute isosceles triangle the centers of the inscribed and circumscribed circles both lie on the altitude to the base.

Ex. 7. In ex. 6, show that if the vertex angle is less than either base angle, the center of the circumscribed circle will be nearer the vertex.

Ex. 8. In ex. 6, suppose that the vertex angle is greater than either base angle.

Ex. 9. Prove that every triangle has three ex-centers. (See ex. 5, p. 111.)

Ex. 10. Prove that the line joining an ex-center of a triangle to the remote vertex of a triangle bisects the interior angle at that vertex.

Proposition XXIII. Problem

298. *Upon a given straight line as a chord to construct a segment of a circle in which a given angle can be inscribed.*

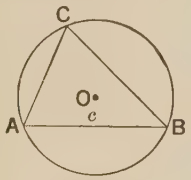
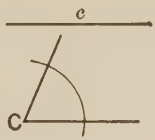


Fig. 1



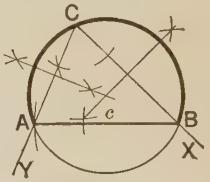


Fig. 2

Given: Line c and $\angle C$.

Required: To construct, with c as chord, a segment of a circle in which $\angle C$ can be inscribed.

Analysis:

1. The drawing when made will appear as in Fig. 1.
2. The problem then is to construct a $\triangle$ like $\triangle ABC$ and circumscribe a circle about it.
3. What parts in this $\triangle$ are given?
4. Show that a $\triangle$ can be constructed with these parts by making $\angle XCY = \angle C$ and side $AB = c$ (Fig. 2).

Construction and Proof:

Left to the pupil.

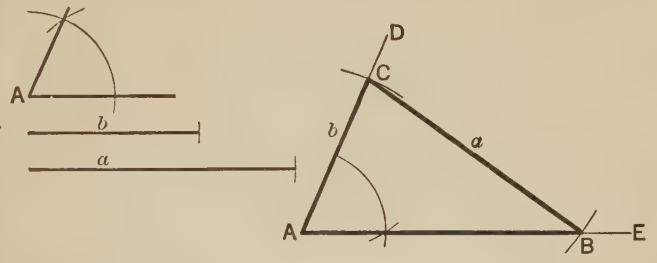
Discussion:

1. Will $\angle C$ remain the same in size if C is taken anywhere in arc ACB?
2. Will $\angle A + \angle B$ remain the same for these positions of C?
3. In Fig. 2 in making $BA = c$ could B be taken *any* point in CX?
4. Is there any restriction on the length of c? on the size of $\angle C$?
5. Could $\triangle ABC$ be isosceles? right? In each case, how would AB in Fig. 2 be constructed?

Ex. Construct Problem XXIII by the use of § 285.

Proposition XXIV. Problem

299. *To construct a triangle, given two sides and the angle opposite one of them.*



Given: a, b, and $\angle A$.

Required: To construct a triangle which shall have these parts.

Construction:

There are several cases, depending on the size of $\angle A$ and the relative lengths of a and b. The method used here, however, applies to all cases.

1. On any line AD take $AC = b$.	1. Post. 3
2. At A construct $\angle CAE = \angle A$.	2. § 74
3. With C as center and a radius equal to a, describe an arc cutting AE at B.	3. Post. 4
4. Join C and B.	4. Post. 1
5. Then $\triangle ABC$ is the required $\triangle$.	

Proof:

Show that $\triangle ABC$ has in it the required parts.

Discussion:

I. When $\angle A$ is a right angle.

1. Can $a = b$? Can a be less than b? greater than b? Why? (See § 164.)
2. Then what relation between a and b makes the problem possible? impossible?
3. How many solutions will there be?

II. When $\angle A$ is an obtuse angle.
Same questions as under I.

III. When $\angle A$ is an acute angle.
1. Can $a = b$? Can a be less than b? greater than b? Why?
2. Then when is the problem possible? (See IV below.)
3. Which of the three possibilities in step 1 give one solution? two solutions?

The pupil should make suitable drawings for each case.

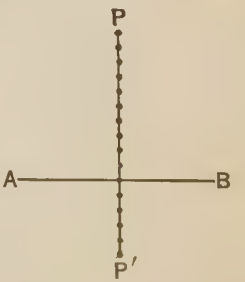
IV. 1. In determining point B, in any case, what is the least length that line a may have?
2. Under what conditions, when $\angle A$ is acute, would the $\triangle$ be a rt. $\triangle$?

This problem, because of the number of possibilities which it presents, is often referred to as the *ambiguous case*.

LOCI

300. If a line is bisected, this bisection point is manifestly the same distance from either end of the line. It is evident, however, that there are many other points in the plane, each of which is also the same distance from either end of the line. If now *just* those points which answer this description could be grouped by themselves apart from all the other points on the paper, we should have what is termed a *locus*.

Thus, if AB is a line and if all points on the paper which are the same distance from A and B are grouped by themselves, this group will form the unlimited line PP' (the perpendicular bisector of AB) and will therefore be the locus of all such points.



301. Two very essential characteristics of this group should be noted. First, its *exclusiveness* — it contains *only* those

points which answer the description. Second, its *inclusiveness* — it contains *all* the points which answer the description. We may therefore define a *locus* in geometry as *the place where are found all points that satisfy a certain condition, and no other points.* In the illustration on page 182 the condition was that each point must be the same distance from A and B.

302. To have a locus, therefore, we must have some condition which each point upon it must satisfy. As soon as this particular condition is known, the nature of the locus is determined. A locus (plural, loci) in geometry may consist of a point or points, a line, or a group of lines.

303. Since the locus must contain all points that satisfy the given condition, it will prove helpful in discovering what a locus is to locate several points (the more the better) each of which satisfies the given condition, and note, if possible, in what particular place they are found.

Exercises — Group 59

Using the above suggestion, determine the locus in each of the following exercises. In each case represent the locus and state the condition satisfied by each "point" on the locus. No proof or discussion is required.

1. Where do you find all points in the front wall of your classroom that are 3′ from the floor?

2. Where do you find all points in two side walls of your classroom that are 5′ from their intersection?

3. What line is the locus of all points in the floor of your classroom that are equidistant from two opposite walls? from two adjacent walls?

4. Where are located all cities that are 3 miles from a given railroad?

5. A man rows a boat up a stream, keeping the same distance from either shore. What is the path (locus) described by the boat?

6. What is the locus of the center of a trolley-car wheel as the car moves along the track?

7. Where are located all houses that are 2 miles from your school building?

8. Where are located all towns that are 20 miles from your town?

9. What is the locus of the end point of the hour hand of a clock as the hand moves around the dial?

10. A five-cent piece is moved around a half dollar, the two constantly touching each other and remaining in the same plane. What is the locus of the center of the five-cent piece?

11. A foot rule lies on a rectangular desk. Find the locus of its mid-point as it slides around if its ends are always on two adjacent edges of the desk.

SIMPLE LOCI

Determining a Locus

304. There are three steps in determining a simple locus:

1. *Locate on the paper several scattered points which satisfy the given condition, and thereby get a notion of what the locus is.*

2. *Prove that every point on this assumed locus satisfies the given condition.*

3. *Prove that every point that satisfies the given condition is on this assumed locus.*

The fact that *both* 2 and 3 are necessary in establishing a locus is seen from the following consideration:

Step 2, taken alone, proves merely that every point on the locus satisfies the given condition. It does not consider the possibility of there being other points off the locus which also satisfy the given condition. It therefore establishes the exclusive, but not the inclusive, characteristic of a locus.

Step 3, taken alone, proves merely that every point that satisfies the condition is on the locus. It does not consider the possibility of there being other points on the locus that do not satisfy the given condition. It therefore establishes the inclusive, but not the exclusive, characteristic of a locus.

As both of these characteristics must be established in proving a locus, both step 2 and step 3 are necessary.

Fundamental Loci Theorems

305. Theorem I. *The locus of points equidistant from two given points is the perpendicular bisector of the line joining the two points.*

See § 203.

306. Theorem II. *The locus of points within an angle equidistant from the sides is the line that bisects the angle.*

See § 205.

307. Corollary. *The locus of points equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by the given lines.*

308. Theorem III. *The locus of points equidistant from two parallel lines is the line parallel to each of the given lines and midway between them.*

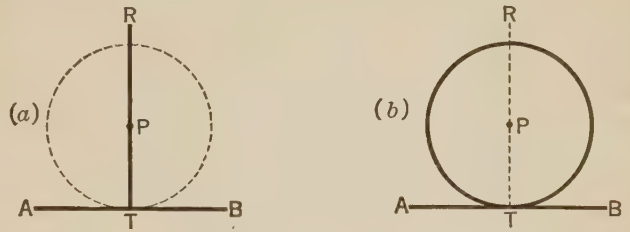
309. Theorem IV. *The locus of points a given distance from a given line consists of two lines, one on either side of the given line, each parallel to the given line and the given distance from it.* (Converse of Th. III.)

310. Theorem V. *The locus of points a given distance from a given point is the circle described with the given point as center and the given line as radius.*

See § 223, 4.

Theorem VI

311. *The locus of the centers of all circles tangent to a given line at a given point is the perpendicular to the line at that point.*



(*a*) **Given**: *P any* point on the locus.
To prove: *P* satisfies the given condition, that is, *P* is the center of a circle to which *AB* is tangent at *T*.

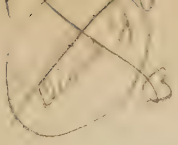
(*b*) **Given**: *P any* point satisfying the condition, that is, the center of a circle tangent to *AB* at *T*.
To prove: *P* is on the locus, that is, *P* is on the ⊥ to *AB* at *T*.

(*a*) **Analysis**:
See §§ 246 and 122.

(*b*) **Analysis**:
See §§ 245 and 92.

The simple loci determined in Theorems I–VI should be carefully worked out and thoroughly understood. Nearly all other loci problems will be found to be applications of one or more of the above theorems.

312. Unless otherwise stated, the *distance of a point from a circle* will mean the shortest distance from the point to the circle (§ 174), and hence it will be measured on a straight line through the center of the circle. (See ex. 3 and 4, p. 166.)



Exercises — Group 60

SIMPLE LOCI

In the following exercises determine the locus, basing it, when possible, on some of the theorems on pages 185 and 186. Read § 303.

1. Determine the locus of all points within a circle that are equidistant from the ends of a given chord.

2. Determine the locus of the vertex of all isosceles triangles constructed on a given line as a base.

3. What is the locus of the mid-points of all radii of a given circle?

4. Determine the locus of the mid-points of equal chords of a given circle.

5. Determine the locus of all points a given distance from a given circle. (See § 312.) When would there be one solution? two solutions?

6. Determine the locus of the centers of all circles passing through two given points. (See § 236.)

* **7.** Determine the locus of the vertex of the right angle of all right triangles constructed on a given line as the hypotenuse.

8. What is the locus of the centers of all circles of a given radius which touch a given line?

9. What is the locus of the vertex of all triangles constructed on a given line as a base and having a given altitude?

10. Determine the locus of the centers of all circles (*a*) tangent to two given parallel lines; (*b*) tangent to two intersecting lines.

313. We have thus far considered a locus as an assemblage of points each of which satisfies some given condition. It is often expedient to consider a locus as *the path traced by a point moving under some restriction*, the restriction always being the same as the " given condition " mentioned above.

These two definitions give the same figure in any locus problem. For example, determine the locus of all points a given distance from a given point. Considered as an assemblage of all points that satisfy

this condition, we get a circle whose center is the given point and whose radius is the given distance. (§ 310.) Considered as the path traced by a point moving so that it is always the given distance from the given point, we obtain, as before, a circle. See ex. 5, 6, 9, 10, p. 184, for other illustrations.

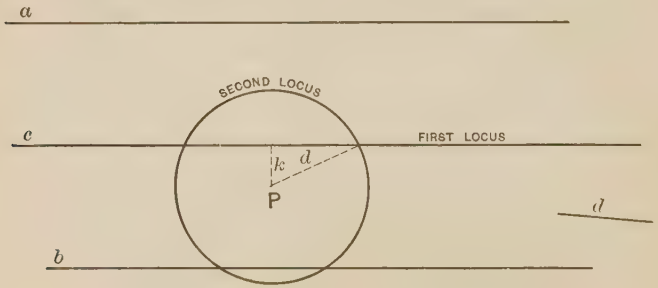
In drawing a circle with compasses, the latter definition is used.

INTERSECTING LOCI

314. Sometimes the points in question are required to satisfy two or more conditions. In such cases they will be the intersections of the separate loci determined by the given conditions. Hence, in solving problems that consist of two or more loci, the locus for each condition is constructed and the intersections of these loci are noted. These intersections, that is, the points which the loci have in common, will be the required locus.

315. Sometimes these loci have no intersection or one intersection or more than one intersection, depending on the size and position of the parts. In solving these problems, take the parts in the most general position, thereby determining the most general locus. Then, by way of discussion, take restricted parts in special positions and determine what the locus would be in these exceptional cases.

Ex. Locate all points that are equidistant from two parallel lines a and b, and a distance d from a given point P.



Discussion:

1. Apply §§ 308 and 310. The number of solutions depends upon the comparative lengths of d and k (see drawing).
2. Suppose that $d > k$; $d = k$; $d < k$.
3. Hence, when is the solution possible? When is it impossible?

Pupils should not misunderstand the use of the word *equidistant* when it occurs twice in any locus problem.

Thus, in ex. 1 below the distances from any point on the locus to the two sides of the angle are to be equal and also the distances from this point to the two given points are to be equal; but *this does not mean that these four distances are all to be equal respectively.*

In solving the following loci problems be sure that all lines are extended indefinitely so that *all* points belonging to the locus are found.

Exercises — Group 61

INTERSECTING LOCI

1. Locate all points within a given angle that are equidistant from the sides of the angle and also equidistant from two given points in one side of the angle. How many points will there be?

2. Locate all points within a given angle that are a given distance from the vertex and also equidistant from the two sides.

3. Find a point at a given distance from two intersecting lines.

4. Locate all points that are equidistant from two given parallel lines and also equidistant from two given points in one of the lines.

5. Locate all points that are a given distance from a given line and also equidistant from two given points on the line.

6. Find all points that are equidistant from two given points and also equidistant from two given parallel lines.

7. Find all points that are a given distance from a given line and also equidistant from two given points.

8. Find all points that are equidistant from two given parallel lines and also a given distance from a given line.

9. Locate all points that are equidistant from two given points and also equidistant from the sides of a given angle.

GENERAL DIRECTIONS FOR SOLVING CONSTRUCTION PROBLEMS

316. In order to construct any geometric figure, enough parts must be given to determine the figure. (See § 68.) However, it is not always apparent how to start the construction when these parts are given. The following suggestions (of which No. 3 is the most important) are helpful in discovering where the start should be made and in completing the construction.

1. *Make a reasonably accurate drawing of the construction as it will appear when completed.*

2. *Indicate on the drawing the parts that are given and any additional parts that may be determined from the given parts by using known properties of the figure that is being constructed.*

3. *Find some part of the figure that is fully determined and start the actual construction with this part. This part is usually a triangle.*

4. *Complete the figure and show that it satisfies the given conditions.*

5. *Discuss any limitations there may be to the solution and the number of possible solutions, if there is more than one.*

317. In the following (and future) constructions the parts of a triangle will always be denoted in the same manner, the sides by a, b, c, the opposite angles by A, B, C, the altitudes by h_a, h_b, h_c, the medians by m_a, m_b, m_c, the bisectors of the angles by t_a, t_b, t_c, and the radii of the circumscribed and inscribed circles by R and r respectively.

Ex. 1. Construct the triangle ABC, given $a + b$, c, and $\angle B$.

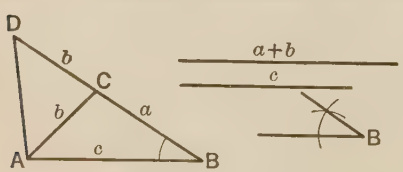


FIG. 1

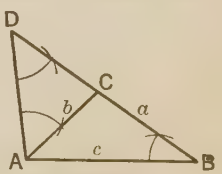


FIG. 2

Analysis:

1. The drawing when made will appear as in Fig. 1.
2. In $\triangle ABD$, AB, $\angle B$, and BD are known.

Construction:

1. Construct $\triangle ABD$ (Fig. 2). (See § 79.)
2. Construct $\angle DAC = \angle D$.
3. Then $\triangle ABC$ is the required $\triangle$.

Proof:

Show that $AB = c$, $\angle B = \angle B$, $BC + CA = BD = a + b$.

Discussion:

This problem is possible when $(a + b) > c$ and $\angle B < 180°$.

Ex. 2. Construct the triangle ABC, given a, c, and m_b.

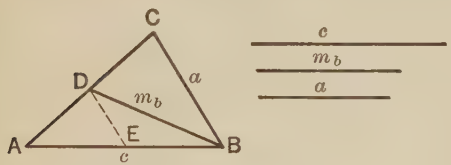


FIG. 1

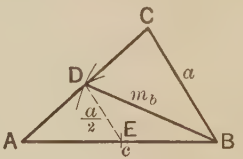


FIG. 2

Analysis:

1. The drawing when made will appear as in Fig. 1.
2. Draw $DE \parallel CB$.
3. Then the sides of $\triangle EBD$ are known. Why?

Construction:

1. Bisect lines a and c.
2. Construct $\triangle EBD$ (Fig. 2). (See § 77.)
3. Extend BE its own length to A and join A to D.
4. Extend AD its own length to C and join C to B.
5. Then $\triangle ABC$ is the required $\triangle$.

Proof:

Show that $\triangle ABC$ has the given parts.

Discussion:

This problem is possible when $(DE + EB) > m_b$; that is, when $(a + c) > 2\,m_b$.

Intersections of loci often facilitate the construction.

Ex. 3. Construct the triangle ABC, given c, h_c, and m_c.

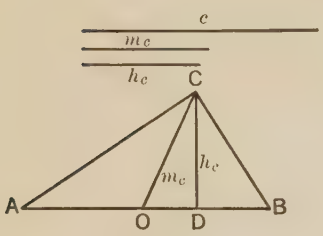


FIG. 1

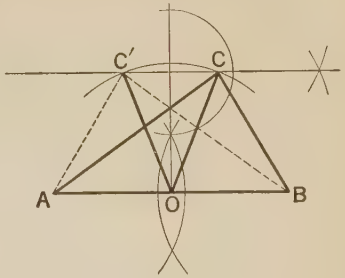


FIG. 2

Analysis:

1. The drawing when made will appear as in Fig. 1.
2. If point C can be located, the problem is solved. Why?
3. How far is C from AB? Then on what locus must it be?
4. How far is C from O? Then on what other locus must it be?
5. Hence where must C be located?

Construction and Proof:

Left to the pupil.

Discussion:

1. There are really four $\triangle$, all congruent. Where are they?
2. The cons. is possible when the two loci meet. What must be the relation between m_c and h_c? Suppose that $m_c = h_c$.

Ex. 4. Construct the parallelogram $ABCD$, given AB, BC, and angle AOB between the diagonals.

Analysis:

1. If O can be located, the problem is solved. Explain.
2. In $\triangle OAB$ draw median OM. OM is known. Why?
3. On what loci is O, by §§ 298 and 310?

Construction and Proof:

Left to the pupil.

Discussion:

The problem is possible when $\angle AOB <$ a st. $\angle$. Explain.

Exercises — Group 62

CONSTRUCTION OF TRIANGLES

Construct a right triangle having given:

* **1.** The two legs.

* **2.** A leg and the adjacent acute angle.

* **3.** A leg and the opposite acute angle.

* **4.** The hypotenuse and either acute angle. (See § 282.)

* **5.** The hypotenuse and either leg.

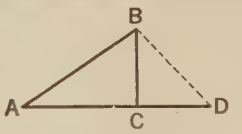
6. The hypotenuse and the altitude on the hypotenuse.

7. An acute angle and the altitude on the hypotenuse.

8. One leg and the radius of the circumscribed circle.

* **9.** An acute angle and the radius of the inscribed circle. (See § 252.)

* **10.** The hypotenuse and sum of the legs.

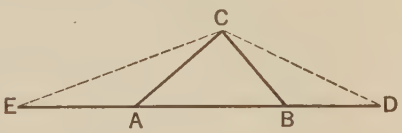


SUG. In $\triangle ADB$, AB, AD, and $\angle D$ are known.

* **11.** An acute angle and the sum of the legs.

SUG. In $\triangle ADB$ (ex. 10), $\angle A$, AD, and $\angle D$ are known.

* **12.** The sum of the sides and an acute angle.



SUG. In $\triangle EDC$, $\angle E$, ED, and $\angle D$ are known.

Construct an isosceles triangle having given:

* **13.** The base and one of the legs.

* **14.** The base and one of the base angles.

* **15.** The base and the vertex angle.

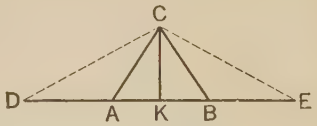
* **16.** The base and the altitude on the base.

17. The altitude on the base and (*a*) one leg (see ex. 5); (*b*) a base angle; (*c*) the vertex angle (see ex. 2).

18. The base and the altitude on one leg. (See ex. 5.)

* **19.** One leg and the altitude on the leg.

* **20.** The sum of the sides and the altitude on the base.



SUG. In $\triangle DEC$, DE and CK are known.

Construct an equilateral triangle having given:

21. The sum of the sides.

22. The altitude.

23. The radius of the inscribed circle.

Construct any triangle having given:

24. a, m_a, C

25. a, b, h_b

26. a, b, h_c

27. h_a, B, C

28. c, h_a, h_b

* **29.** c, C, m_c (See § 298.)

30. a, A, h_a

31. C and segments of c made by h_c

32. a, B, r (See ex. 9.)

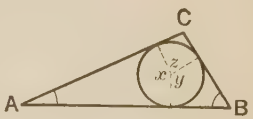
* **33.** $a + b + c$, A, B (See ex. 12.)

34. a, m_b, m_c

35. A, $c + a$, h_c

36. a, C, $b - c$

37. A, B, r



SUG. $\measuredangle$ x, y, and z are known.

Exercises — Group 63

CONSTRUCTION OF QUADRILATERALS

Construct a square having given:

1. The sum of the four sides.

2. A diagonal.

Construct a rectangle having given:

3. A diagonal and the angle between the diagonals.

4. A diagonal and the sum of the four sides. (See ex. 10, p. 193.)

Construct a rhombus having given:

* **5.** The two diagonals.

6. One side and the altitude on that side.

7. One angle and a diagonal.

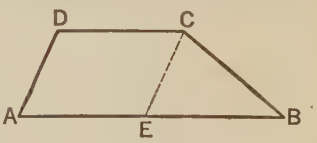
Construct any parallelogram having given:

8. One side and the two diagonals.

9. One side, one angle, and one diagonal.

10. An angle, one side, and the altitude on that side.

NOTE. In constructing a trapezoid it will often be helpful to observe that it may be divided into a parallelogram and a triangle by a line through a vertex parallel to the opposite leg. In the drawing, $CE = DA$, $EB = AB - DC$, and $\angle BEC = \angle A$. Hence the construction is often effected by starting with triangle EBC. (See § 316, 3.)



Construct an isosceles trapezoid having given:

11. The bases and an angle. (See ex. 14, p. 193.)

12. The bases and the altitude.

13. The bases and one diagonal.

SUG. In the drawing, AC and AF are known.

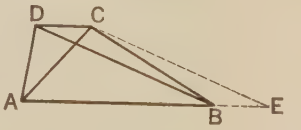
Construct any trapezoid having given:

* **14.** The four sides.

15. The bases and diagonals. (See ex. 13; also ex. 13, p. 104.)

16. A base, the diagonals, and the angle between the diagonals.

SUG. AC, $\angle ACE$, and CE are known.



Construct any quadrilateral having given:

17. The four sides and the angle between two adjacent sides.

Exercises — Group 64

CONSTRUCTION OF CIRCLES

Construct a circle (see § 316, 5) *that:*

1. Passes through two given points and has its center in a given line.

2. Touches a given line at a given point and has its center in another given line.

3. Has a given radius and passes through two given points.

4. Has a given radius and touches a given circle at a given point.

5. Touches two given lines, one of them at a given point.

6. Touches two parallel lines and passes through a given point.

7. Has a given radius, touches a given line, and also touches a given circle.

8. Has a given radius, touches a given line, and passes through a given point.

* **9.** Has a given radius and touches two intersecting lines.

10. Touches a given line and a given circle at a given point.

SUG. Draw the tangent at the given point and use §§ 311 and 306.

Exercises — Group 65

MISCELLANEOUS CONSTRUCTIONS

1. Construct a chord of a given circle, having given the midpoint of the chord.

2. From a given point in a circle draw a chord that shall be a given distance from the center.

3. Through a given point within a circle draw a chord equal to a given chord. (See ex. 5, p. 193.)

4. Draw a line touching a given circle and parallel to a given line. (See §§ 139 and 246.)

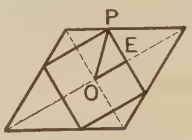
5. Within a given circle draw a chord equal to and perpendicular to a given chord.

6. Through a given point draw a line making equal angles with two given lines. (See ex. 6, p. 67.)

7. Through a given point draw a line cutting two given parallel lines so that the segment intercepted between the parallels shall be of a given length. Is there any restriction on the given length? (See § 292.)

8. Given two lines that would meet if prolonged. Construct the bisector of the angle between them without producing the lines.

9. Inscribe a square in a given rhombus.



Sug. $\triangle OEP$ is a rt. isosceles $\triangle$. Hence locate point P.

* **10.** Construct a common external tangent to two given circles.

Sug. See Fig. 1 below. In trapezoid $OO'T'T$, OT, OO', $O'T'$ and $\angle T$ are known. See Note on cons. of trapezoid, p. 195; also ex. 5, p. 193. Start the cons. with $\triangle ORO'$.

* **11.** Construct a common internal tangent to two given circles.

Sug. See Fig. 2 below. Start the cons. with $\triangle ORO'$.

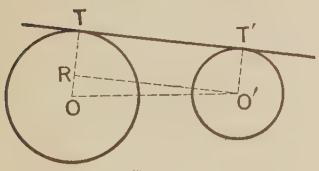


Fig. 1

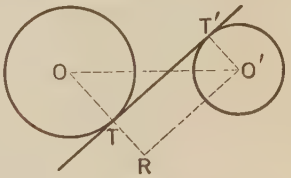


Fig. 2

Exercises — Group 66

Miscellaneous Loci

1. In a given line find a point that shall be (*a*) equidistant from two given points; (*b*) a given distance from a given point.

2. Locate all points that are equidistant from the sides of a given angle and also equidistant from two given parallel lines.

3. Locate all points that are a given distance from a given line and also a given distance from a given point.

4. Find a point in one side of a given triangle that is equidistant from the other two sides.

5. $ABCD$ is a given parallelogram. Find a point in AB that is equidistant from BC and CD.

6. Find the locus of the mid-points of all lines drawn from a given point to a given line.

7. Find a point within a given triangle which is equidistant from two sides of the triangle and a given distance from the third side.

8. What is the locus of all points equidistant from two given concentric circles? (See § 312.)

9. Find all points at given distances a and b from two given points. Discuss.

10. Determine the locus of the centers of all circles that touch a given circle at a given point. (See § 311.)

11. Find the locus of the end points of tangents of a given length drawn to a given circle.

***12.** Two given circles are tangent to each other. Find the locus of all points from which tangents to the two circles are equal.

13. Given the base of a triangle and the median to the base. Find the locus of the end of the median remote from the base.

14. What is the locus of the center of a given circle as it rolls around the sides of a square, always remaining in the plane of the square?

15. Find the locus of the intersection of the diagonals of all rectangles constructed on a given line AB as a base.

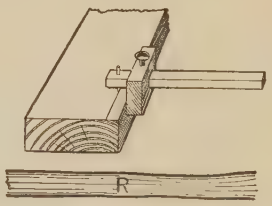
16. On AB as a base, triangles are constructed, each having one of the two remaining sides of a given length. Find the locus of the mid-point of the third side.

17. Find a point on a given circle that is equidistant from the sides of all angles inscribed in a given segment of that circle.

18. On a given line AB parallelograms are constructed. Find the locus of the intersection of the bisectors of angles A and B.

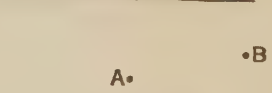
Exercises — Group 67

Applications



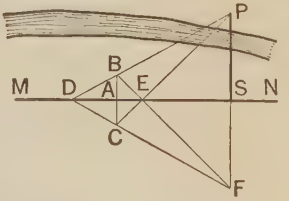
1. What geometric principle is involved in the use of a carpenter's marking gauge?

2. Show where a long-range gun should be located on road R so that its range to each of two supply stations, A and B, will be the same.



3. A robber buried some loot in a field between two intersecting roads, 50 ft. from one road and 30 ft. from a tree which stood by the other road. Show how to locate the hiding place.

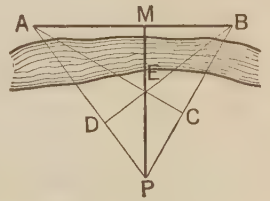
4. To set out a perpendicular to a given line MN from a given inaccessible point P.



At any point A in MN set out a perpendicular AB of any convenient length. Prolong it on the other side of MN the same distance AC. Mark on MN a point D in line of PB and a point E in line of PC. Mark the point F at the intersection of DC and BE prolonged. The line FP is the required line.

Prove that this method is correct.

5. To set out a perpendicular from a given point P to a given inaccessible line AB.

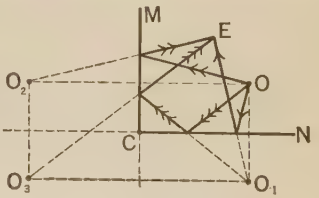


Let fall perpendiculars from A and B to BP and AP respectively at C and D. The line PE, passing through the given point and the intersection of these perpendiculars, is the required perpendicular to AB.

Prove that this method is correct.

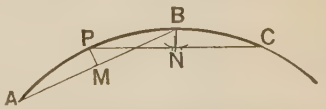
6. Two nails A and B are driven in a board, and a carpenter's square is placed flat on the board so that its two edges are always against the two nails. What is the locus of the *heel* C of the square as the square moves about in different positions?

7. Show that when two plane mirrors have their reflecting surfaces at right angles, an object placed between them presents *three* images all situated in a circle whose center is on the intersection of the two mirrors and whose radius is the distance from the intersection to the object.



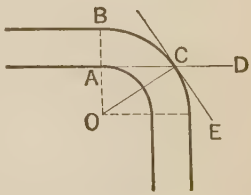
8. To extend a curve AB which is known to be an arc of a circle when the center is inconveniently located or the radius is too large to swing arcs with a tape.

Find M, the mid-point of chord AB, and run a perpendicular from M to P. Draw an arc with B as center and PM as radius, and a second arc with P as center and AM as radius, intersecting the first arc in N. Extend PN its own length to C and thereby locate point C on the circle. Other points may be located in a similar way.



Prove that this method is correct.

9. A street railway track constructed around a corner to connect two straight tracks is usually made in the form of an arc of a circle tangent to each of the two straight tracks at their end points. Show how to find the radius of the outer rail (*radius of the turnout*) and the center of the circle.



10. In the case of a simple turnout (see ex. 9) the angle between the straight track AD and the tangent to the track at C, that is, angle ECD, is called the *angle of the turnout*. Show that this angle equals angle COB.

BOOK THREE

PROPORTIONS. SIMILAR POLYGONS

PROPORTIONAL QUANTITIES

318. The *ratio* of line a to line b is the quotient or indicated quotient obtained by dividing the numerical measure of a by the numerical measure of b. It is written $\frac{a}{b}$ or $a : b$ and means "a divided by b."

319. It follows that all ratios are common fractions and, conversely, any common fraction may be regarded as a ratio.

The numerator or first term of a ratio is called the *antecedent;* the denominator or second term is called the *consequent.*

320. Since the terms of a ratio may be regarded as the terms of a common fraction, it follows that:

The value of a ratio remains the same if both of its terms are multiplied or divided by the same number.

Ex. 1. Does the value of a ratio remain the same if we add the same number to, or subtract it from, both its terms? Illustrate.

Ex. 2. Express in simplest form the ratio $a : b$ when $a = 6''$, $b = 8''$; $a = 15''$, $b = 3'$; $a = 2\frac{1}{2}''$, $b = 6\frac{1}{4}''$; $a = 8\,m$, $b = 12\,n$.

Ex. 3. Cut a line $4\frac{7}{8}''$ long into two parts in the ratio 3 : 10.

321. A *proportion* is an expression of equality between two ratios. The four quantities in the proportion are the *terms of the proportion.* The first and fourth terms are called the *extremes;* the second and third terms are called the *means.*

Thus, if the ratios $a : b$ and $c : d$ are equal, the proportion is $\frac{a}{b} = \frac{c}{d}$ or $a : b = c : d$. In each case it is read "a is to b as c is to d." Here a, b, c, d are the terms of the proportion, a and d are the extremes, and b and c are the means.

322. It follows at once that all proportions are fractional equations and hence that *any rule or axiom applicable to equations becomes at once applicable to proportions.*

323. It must constantly be kept in mind that in any proportion involving lines, the terms are really numbers which represent the numerical measures of the lines. In any other sense such proportions would be meaningless.

Ex. 1. Determine x if each of the following is to be a proportion:

(*a*) $\frac{3}{4} = \frac{12}{x}$ (*c*) $6:(5-x) = 7:3$ (*e*) $m:n = (s-x):r$

(*b*) $7:10 = x:5$ (*d*) $\frac{(x-6)}{4} = \frac{4}{5}$ (*f*) $\frac{x-2}{2} = \frac{x}{3x-3}$

Ex. 2. In each of the following determine whether the four quantities, taken in the order given, form a proportion:

(*a*) 6, 8, 3, 4 (*c*) $2\frac{1}{2}$, 4, 10, 16 (*e*) 6, $2\frac{5}{8}$, 16, 7

(*b*) 5, .3, 20, 1.2 (*d*) 3 in., $2\frac{1}{2}$ ft., 1 ft., $3\frac{1}{3}$ yd. (*f*) 8, .6, 5, .4

Ex. 3. When measuring four quantities to determine whether they are proportional, is it necessary to use the same unit of measure throughout? Illustrate.

324. The *fourth proportional* to three given quantities is the fourth term of the proportion whose first three terms are the three quantities taken in order.

Thus, in the proportion $a:b = c:d$, d is the fourth proportional to a, b, and c.

325. In any proportion in which the two means are equal, either mean is said to be the *mean proportional* between the first and last terms. In the same proportion the last term is said to be the *third proportional* to the first and second terms.

Thus, in the proportion, $a:b = b:c$, b is the mean proportional between a and c, and c is the third proportional to a and b.

326. The quantities a, b, c, d . . . are said to be in *continued proportion* if $a:b = b:c = c:d$ . . .

Ex. 1. If x is the fourth proportional to each of the following groups of numbers, write the proportion in each case and find x:

3, 4, 9; 3, 9, 4; 4, 9, 3; 4, 3, 9; 9, 3, 4; 9, 4, 3

Ex. 2. Is the fourth proportional to three given quantities the same irrespective of the order in which they are taken?

Ex. 3. If x is the mean proportional between each of the following pairs of numbers, write the proportions and find x:

5, 20; 20, 5; 8, 18; 18, 8; 8, 6; 6, 8; a, b; b, a

Ex. 4. Is the mean proportional between two given quantities the same irrespective of the order in which they are taken?

Ex. 5. If x is the third proportional to each of the following pairs of numbers, write the proportion in each case and find x:

3, 6; 6, 3; 8, 12; 12, 8; r, s; s, r

Ex. 6. In ex. 5, can the quantities be taken in either order?

FUNDAMENTAL PROPERTIES OF PROPORTIONS

327. PROPERTY 1. *In any proportion the product of the means equals the product of the extremes.*

Given: $\frac{a}{b} = \frac{c}{d}$.

To prove: $a \times d = b \times c$.

See § 322. What axiom is used?

This may be called *the* fundamental property of a proportion.

328. COROLLARY. *If three terms of one proportion are equal respectively to the three corresponding terms of another proportion, the fourth terms are also equal.*

329. PROPERTY 2. *If the product of two quantities is equal to the product of two other quantities, either pair may be used as the means and the other pair as the extremes of a proportion.*

Given: $a \times d = b \times c$.

To prove: $\frac{a}{b} = \frac{c}{d}$.

This is the converse of § 327. Hence what axiom should be used?

330. PROPERTY 3. *The mean proportional between two quantities is equal to the square root of their product.*

Given: $a : b = b : c$.

To prove: $b = \sqrt{a \times c}$.

See § 327.

331. PROPERTY 4. *The products of the corresponding terms of two or more proportions are in proportion.*

Given: $\frac{a}{b} = \frac{c}{d}$ and $\frac{m}{n} = \frac{p}{q}$.

To prove: $\frac{a \times m}{b \times n} = \frac{c \times p}{d \times q}$.

What axiom justifies this operation?

332. PROPERTY 5. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Given: $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$

To prove: $\frac{a + c + e}{b + d + f} = \frac{a}{b}$.

Analysis:

1. The required proportion is true if
   $ab + cb + eb = ab + ad + af$. Why?
2. The statement in 1 is true if the terms on the two sides are equal respectively. Are they? Explain.
3. Hence, starting with the three equalities found in 2, obtain step 1, and hence the required proportion.

Discussion:

Show that $\frac{a + c + e}{b + d + f} = \frac{c}{d} = \frac{e}{f}$.

Exercises — Group 68

1. *Find x in each of the following proportions:*

(*a*) $5:8 = 4:x$ (*d*) $x:8 = 12:16$

(*b*) $3:x = 7:4$ (*e*) $1\frac{1}{3}$ ft. : 30 in. = 4 yd. : x ft.

(*c*) $6:9 = x:12$ (*f*) a ft. : b in. = x ft. : c in.

* **2.** If $\frac{r}{s} = \frac{t}{x}$ and $\frac{r}{s} = \frac{t}{y}$, prove that $x = y$.

* **3.** If $a:x = a:y$, prove that $x = y$.

4. Form all possible proportions from each of the following products: (*a*) $3 \times 16 = 8 \times 6$; (*b*) $r \times s = p \times q$.

5. Form all possible proportions from $10 = ax$ in which a occurs as a mean. (Use only integers.)

6. Find the mean proportional between each of the following pairs of numbers:

(*a*) 3, 12 (*c*) 12, 9 (*e*) $10\frac{2}{3}''$, 8′

(*b*) 7, 28 (*d*) 2′, 6′ (*f*) x, y

7. If $\frac{a}{b} = \frac{c}{d}$ and $\frac{r}{s} = \frac{m}{n}$, form a single proportion involving the eight quantities.

8. Using § 332, form three different proportions from $\frac{5}{2} = \frac{15}{6} = \frac{20}{8}$.

9. If $\frac{x - r - 3}{y - s - 5} = \frac{r}{s} = \frac{3}{5}$, find the value of $\frac{x}{y}$.

FUNDAMENTAL TRANSFORMATIONS OF PROPORTIONS

333. TRANSFORMATION 1. *If four quantities are in proportion, they are in proportion by* ***alternation****; that is, the first term is to the third term as the second is to the fourth.*

Given: $a:b = c:d$.

To prove: $a:c = b:d$.

Analysis:

1. By § 327 how may the given proportion be written?
2. From this form obtain the desired form by Ax. 5.

334. TRANSFORMATION 2. *If four quantities are in proportion, they are in proportion by* ***inversion****; that is, the second term is to the first term as the fourth is to the third.*

Given: $a : b = c : d$.

To prove: $b : a = d : c$.

Analysis:

Same as in § 333.

335. TRANSFORMATION 3. *If four quantities are in proportion, they are in proportion by* ***addition****; that is, the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.*

Given: $a : b = c : d$.

To prove: $a + b : b = c + d : d$.

Analysis:

Use the method employed in § 332.

336. COROLLARY. *If* $a : b = c : d$, *then* $a + b : a = c + d : c$.

337. TRANSFORMATION 4. *If four quantities are in proportion, they are in proportion by* ***subtraction****; that is, the difference of the first two terms is to the second term as the difference of the last two terms is to the fourth term.*

Given: $a : b = c : d$.

To prove: $a - b : b = c - d : d$.

Analysis:

Same as in § 335.

338. COROLLARY. *If* $a : b = c : d$, *then* $a - b : a = c - d : c$.

The proportion resulting from Trans. 3 is often referred to as proportion by *composition;* the proportion resulting from Trans. 4 is often referred to as proportion by *division.*

339. AXIOM 14. *Like powers or like roots of equal quantities are equal.*

340. TRANSFORMATION 5. *If four quantities are in proportion, like powers or like roots of these quantities are in proportion.*

Given: $a : b = c : d$.

To prove: (1) $a^n : b^n = c^n : d^n$; (2) $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}$.

Analysis:

See Ax. 14, p. 206.

Exercises — Group 69

1. By successive transformations of the proportion $a : b = c : x$, write the proportions in which x occurs as the third, second, and first terms respectively.

2. Write $x = \frac{ab}{c}$ so that x occurs as the fourth term of the corresponding proportion; as the third term of the proportion.

***3.** If $a : b = c : d$, show that $a + b : a - b = c + d : c - d$.

Here a, b, c, and d are said to be in proportion by *addition and subtraction* (sometimes called *composition and division*).

4. Test the validity of the transformations by alternation, by inversion, and by addition and subtraction by transforming each of the following proportions:

(*a*) $12 : 8 = 9 : 6$ (*b*) $.18 : 2.7 = .2 : 3$ (*c*) $2r : rs = 2s : s^2$

5. Find the ratio $a : b$ in each of the following:

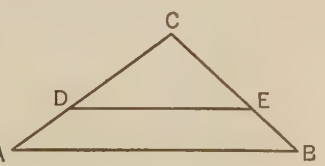
(*a*) $\frac{a+b}{b} = \frac{5}{3}$ (*c*) $\frac{a+b}{a-b} = 7$

(*b*) $a - b : b = 2 : 3$ (*d*) $a^3 : b^3 = 8 : 27$

(*e*) $\sqrt{a} : 2 = \sqrt{b} : 3$

***6.** To what two independent proportions is the series of equal ratios $a : m = b : n = c : r$ equal?

***7.** In the triangle ABC, $AD : DC = BE : EC$.

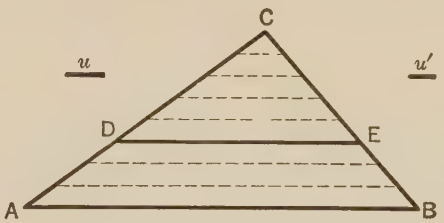


(*a*) Write all proportions that follow by §§ 333, 334.

(*b*) Write simplified all proportions that follow by §§ 335 to 338.

Proposition I. Theorem

341. *If a line is drawn through two sides of a triangle parallel to the third side, it divides the two sides proportionally.*



Given: $\triangle ABC$, $DE \parallel AB$ and cutting AC in D and BC in E.

To prove: $AD : DC = BE : EC$.

Case I. When AD and DC are commensurable. (See § 268.)

Proof:

1. Let u be a unit of measure common to AD and DC, and let u be contained in AD m times and in DC n times. Then $\frac{AD}{DC} = \frac{m(u)}{n(u)} = \frac{m}{n}$.	1. § 318
2. At the points of division on AC draw lines $\parallel$ to AB.	2. § 132
3. These lines divide BE into m parts and EC into n parts, each equal to u'.	3. § 191
4. Then $\frac{BE}{EC} = \frac{m(u')}{n(u')} = \frac{m}{n}$.	4. § 318
5. $\therefore \frac{AD}{DC} = \frac{BE}{EC}$.	5. Ax. 1

Case II. When AD and DC are incommensurable.

The relation is still true. However, the truth of the statement in this case will be assumed here.

342. Other forms immediately derived from the proportion in § 341 are:

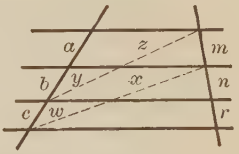
(1) $DC : AD = EC : BE$. Why?
(2) $AD : BE = DC : EC$. Why?

343. COROLLARY 1. *If a line is drawn through two sides of a triangle parallel to the third side, either side is to one of its segments as the other side is to the corresponding segment.*

To prove: (1) $AC : DC = BC : EC$. (See § 335.)
(2) $AC : AD = BC : BE$. (See § 342 (1) and § 335.)

344. COROLLARY 2. *Corresponding segments cut off on two transversals by a series of parallels are proportional.*

To prove: $a : m = b : n = c : r$. (A)



Analysis:

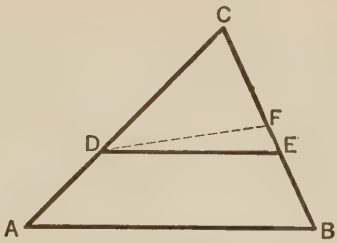
1. This is equivalent to proving
   $a : m = b : n$ and $b : n = c : r$. (B)
   That is, $a : b = m : n$ and $b : c = n : r$. Why? (C)
2. What construction gives a ratio equal to both $a : b$ and $m : n$?
3. What construction gives a ratio equal to both $b : c$ and $n : r$?
4. Then are the proportions true in (C)? in (B)? in (A)?

Discussion:

1. Why is the number of divisions the same in AD and BE?
2. If AD and DC are commensurable, *must* BE and EC be commensurable?
3. If AD and DC are commensurable, would a parallel to AC through E divide AB into commensurable segments? Explain.
4. Does $u = u'$? Discuss.
5. Must the ‖s in Cor. 2 be the same distance apart?
6. In Cor. 2, suppose that the transversals are ‖.
7. Is § 191 a special case of Cor. 2? Explain.

Proposition II. Theorem

345. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.* (Converse of Th. I.)



Given: $\triangle ABC$ and DE drawn so that $AD : DC = BE : EC$.

To prove: $DE \parallel AB$.

Analysis:

1. Suppose DE not $\parallel AB$; draw $DF \parallel AB$ and show that DE and DF coincide.
2. What relation between EC and FC would make DE and DF coincide?
3. By § 343 (1) write a proportion involving EC; FC.
4. Hence show that the required relation in 2 is true by § 328.

Proof:

Left to the pupil.

346. Corollary. *If* $AC : DC = BC : EC$, *then* $DE \parallel AB$.

Exercises — Group 70

1. Show how § 192 may be proved by § 341.

2. In § 341, if $AD = 4''$, $DC = 6''$, $BE = 13''$, find EC.

3. In § 341, if $AC = 15$, $DC = 10$, $EC = 9$, find BE.

4. In § 341, if $DC = a$, $BE = b$, $EC = c$, find AD.

5. In § 344, if $a = 12$, $b = 5$, $c = 15$, $m = 10$, find n and r.

6. Show how § 193 (a) may be proved by § 345.

*** 7.** In § 345, if $AC : AD = BC : BE$, prove $DE \parallel AB$.

8. In § 345, if $AD = 3$, $DC = 2\frac{1}{2}$, $BE = 4$, $EC = 3\frac{1}{3}$, is $DE \parallel AB$?

9. In § 345, if $AD = 1\frac{1}{2}$ ft., $DC = 2$ yd., $BE = 18$ in., $EC = 3$ ft., is $DE \parallel AB$?

10. In § 345, if $AC = 12$, $AD = 9$, $CE = 2$, $EB = 6$, is $DE \parallel AB$?

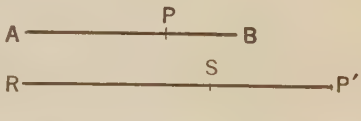
11. In § 345, if $DC = 3\,AD$, $EB = \frac{1}{4}\,BC$, is $DE \parallel AB$?

12. If $\sqrt{x} : 2 = \sqrt{x+3} : \sqrt{5}$, find x.

INTERNAL AND EXTERNAL DIVISION OF A LINE

347. A point is said to divide a given line into segments *internally* or *externally* according as the point is on the line or on the prolongation of the line. In either case one segment is the line from one end of the given line to the point of division, and the other segment is the line from the point of division to the other end of the given line.

Thus, in the drawing, AB is divided internally by point P into segments AP and PB; RS is divided externally by point P' into segments RP' and $P'S$.



Ex. 1. Could the ratio of the two segments be unity (*a*) when the division is internal? (*b*) when the division is external?

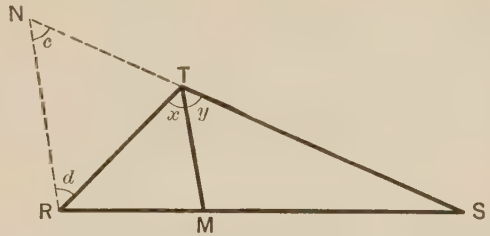
Ex. 2. In either case, could the ratio between the two segments be a proper fraction? an improper fraction? an integer? Illustrate.

Ex. 3. Divide a line 12″ long (*a*) internally; (*b*) externally into two segments whose ratio is 1 : 2; 5 : 3; 7 (that is, 7 : 1).

Ex. 4. A line has been divided into two segments whose ratio is 3 : 2. One of the segments is 6″. Find the length of the line if the division is (*a*) internal; (*b*) external.

Proposition III. Theorem

348. *The bisector of an interior angle of a triangle divides the opposite side internally into segments which have the same ratio as the other two sides.*



Given: $\triangle RST$ and TM the bisector of interior angle RTS.

To prove: $RM : MS = RT : TS$.

Analysis:

1. Th. I is suggested. It will be helpful to draw that triangle resting on side AC.
2. To what does RM correspond in that drawing? MS? TS? MT?
3. How can we get a line to correspond to AB? BE?
4. What proportion is desired? What proportion follows from the drawing? Why?
5. Hence, what lines must be proved equal?
6. What angle equality would make these lines equal?
7. To what $\measuredangle$ are they equal? Why?
8. Hence are the $\measuredangle$ in 6 equal? the lines in 5? the proportion in 4?

Proof:

Left to the pupil.

Ex. 1. In § 348, if $RM = 6$, $MS = 8$, $RT = 10$, find TS.

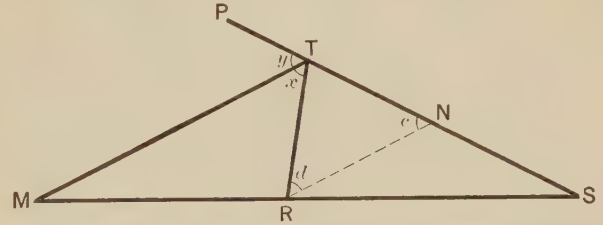
Ex. 2. In § 348, if $RS = 12$, $RT = 6$, $TS = 9$, find RM and MS.

***Ex. 3.** State and prove the converse of Th. III.

Ex. 4. In § 348, if $RM = 3$, $MS = 4$, $RT = 7\frac{1}{2}$, $TS = 9\frac{3}{5}$, does TM bisect $\angle T$?

Proposition IV. Theorem

349. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments which have the same ratio as the other two sides.*



Given: $\triangle RST$ and TM the bisector of exterior angle RTP.

To prove: $RM : MS = RT : TS$.

Analysis:

1. § 343 is suggested. Why?
2. Use the method of Th. III, replacing AB by DE in step 3.

Proof:

Left to the pupil.

Ex. 1. In § 349, if $RS = 12$, $RT = 6$, $TS = 9$, find MR.

Ex. 2. In § 349, if $MR = 9$, $RT = 4\frac{4}{5}$, $TS = 12\frac{4}{5}$, find RS.

Ex. 3. In $\triangle ABC$, $AB = 18$, $BC = 9$, $AC = 12$. If the exterior angle at C is bisected, would the bisector meet the opposite side extended through B or extended through A? Generalize.

Ex. 4. Is Th. IV possible when $TR = TS$? Explain.

* **Ex. 5.** State and prove the converse of Th. IV.

Ex. 6. In § 348, if $RT = a$, $ST = b$, $RS = c$, find RM and MS.

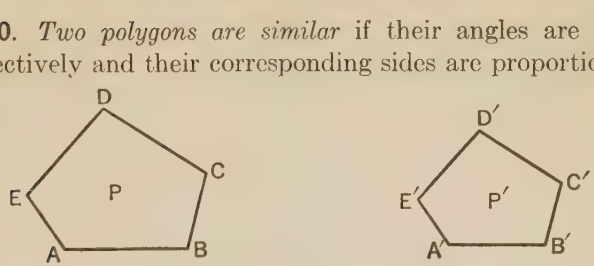
Ex. 7. In § 349, if $RT = s$, $ST = r$, $RS = t$, find RM and MS.

Ex. 8. The sides of a triangle are 17″, 15″, 24″. What point in the longest side must be joined to the opposite vertex to bisect the largest angle? Draw to scale and check.

* **Ex. 9.** Prove that there is but one point which divides a given line internally in a given ratio.

SIMILAR POLYGONS

350. *Two polygons are similar* if their angles are equal respectively and their corresponding sides are proportional.



Thus, in the drawings, P and P' are similar polygons if

(1) $\angle A = \angle A', \angle B = \angle B' \cdots$

(2) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \cdots$

351. *In two similar polygons:*

1. Two equal angles (one in each polygon) are called *corresponding angles.*
2. Two sides (one in each polygon) which join the vertices of two pairs of corresponding angles are called *corresponding sides.*

352. *In the case of similar triangles, corresponding sides will always lie opposite a pair of equal* (that is, *corresponding*) *angles.*

353. From § 350 it follows that for two polygons to be similar, two distinct conditions are necessary:

1. The corresponding angles must be equal.
2. The corresponding sides must be proportional.

For polygons generally, one of these conditions may be true and not the other. Thus, the angles of a rectangle and of a square are equal respectively, but the corresponding sides are not proportional. Also, a rhombus and a square have corresponding sides proportional, but the angles are not equal respectively.

In the case of triangles, however, it will be shown later that if either condition is true, the other *must* also be true. (See § 362.)

354. The ratio of two corresponding sides of two similar polygons is called the *ratio of similitude* of the polygons.

355. Two polygons are *mutually equiangular* if their angles are equal respectively. Two polygons are *mutually equilateral* if their sides are equal respectively.

Exercises — Group 71

1. Are two congruent polygons necessarily similar? Why? If similar, what would be the ratio of similitude?

2. Are any two similar polygons necessarily congruent? Why?

*** 3.** Show that two similar polygons must have the same number of sides.

4. Are any two regular polygons similar? Explain. Would two regular hexagons be similar?

*** 5.** Prove that two similar polygons are congruent if their ratio of similitude is 1.

*** 6.** Prove that any two equilateral triangles are similar.

*** 7.** Prove that any two squares are similar.

8. Prove that any two equiangular triangles are similar.

9. The sides of a polygon are 4, 6, 8, 10, 12. Find the sides of a similar polygon if the ratio of similitude is: $1:2$; $2:3$; $1\frac{1}{2}:1$.

10. A rectangle is $4'' \times 6''$. What are the sides of a similar rectangle if the ratio of similitude is $3:4$?

11. Prove that the line which joins the mid-points of two sides of a triangle cuts off a triangle similar to the given triangle.

Sug. Show by § 193 that both conditions for similarity are true.

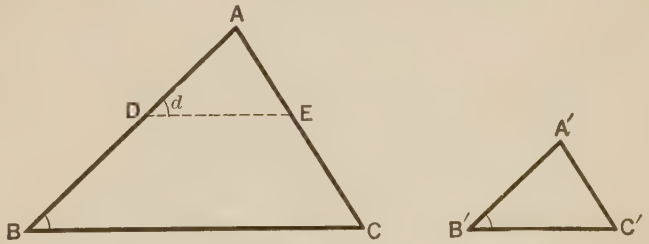
12. Prove that an equilateral quadrilateral inscribed in a circle is regular.

Sug. What condition for similarity remains to be proved?

13. Prove that an equiangular quadrilateral circumscribed about a circle is regular.

Proposition V. Theorem

356. *If two triangles have the three angles of one equal respectively to the three angles of the other, the triangles are similar.*



Given: $\triangle ABC$ and $A'B'C'$, with $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

To prove: $\triangle ABC \sim \triangle A'B'C'$.

Analysis:

1. By def., what remains to be proved to make the $\triangle$ similar?
2. Place the $\triangle$ so that this condition follows, by §§ 138 and 343.

Proof:

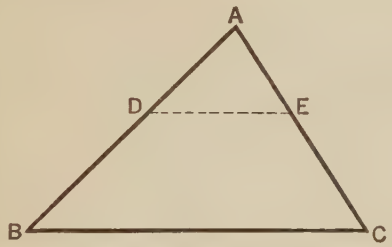
1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$, B' taking the position of D, and C', the position of E.	1. Post. 5
2. Then $\angle d$ ($\angle B'$) $= \angle B$.	2. Hyp.
3. $\therefore DE \parallel BC$.	3. Why?
4. Hence $AB : AD = AC : AE$; that is, $AB : A'B' = AC : A'C'$.	4. § 343
5. Likewise, by placing $\angle B'$ upon $\angle B$, we may prove $BA : B'A' = BC : B'C'$.	5. Reasons 1–4
6. $\therefore \triangle ABC \sim \triangle A'B'C'$.	6. § 350

357. Corollary 1. *If two angles of one triangle are equal respectively to two angles of another, the triangles are similar.*

358. Corollary 2. *Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.*

PROPOSITION VI. THEOREM

359. *If two triangles have an angle of one equal to an angle of the other and the including sides proportional, the triangles are similar.*



Given: $\triangle ABC$ and $A'B'C'$ with $\angle A = \angle A'$ and $AB : A'B' = AC : A'C'$.

To prove: $\triangle ABC \sim \triangle A'B'C'$.

Analysis:

1. If $\triangle A'B'C'$ is placed on $\triangle ABC$ as in the drawing, what relation between BC and DE ($B'C'$) follows by § 346?

2. Then why are the $\triangle$ similar? (See § 357.)

Proof: Left to the pupil.

360. COROLLARY. *If the legs of one right triangle are proportional to the legs of another, the triangles are similar.*

* **Ex. 1.** Two isosceles triangles are similar if any angle of one equals the corresponding angle of the other.

Ex. 2. If $AC = 9$, $AE = 3$, $AB = 12$, $AD = 4$, prove: $\triangle ABC \sim \triangle AED$.

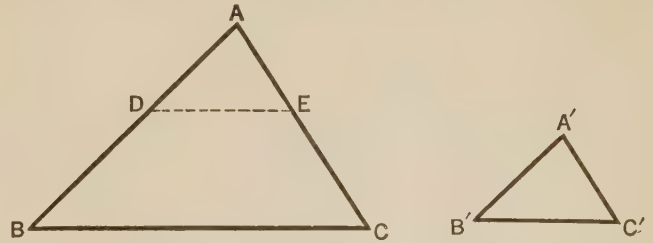
* **Ex. 3.** Prove that if a line is drawn parallel to the base of a triangle, it cuts off a triangle similar to the given triangle.

Ex. 4. In the triangle ABC, AE is the altitude on BC and CD is the altitude on AB.

Prove: $\triangle CDB \sim \triangle ABE$.

Proposition VII. Theorem

361. *If two triangles have their sides respectively proportional, the triangles are similar.*



Given: $\triangle ABC$ and $A'B'C'$, with $AB : A'B' = AC : A'C' = BC : B'C'$.

To prove: $\triangle ABC \sim \triangle A'B'C'$.

Analysis:

1. Form $\triangle ADE$, making $AD = A'B'$ and $AE = A'C'$.
2. Show $\triangle ADE \sim \triangle ABC$, by § 359.
3. Then show $\triangle ADE \cong \triangle A'B'C'$, by § 328 and § 119.

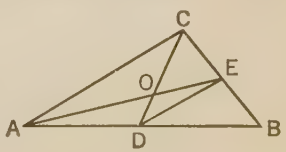
Proof:

1. On AB and AC respectively lay off $AD = A'B'$ and $AE = A'C'$. Then draw DE.	1. Post. 1
2. Then $\triangle ADE \sim \triangle ABC$.	2. § 359
3. $\therefore AB : AD = BC : DE$.	3. By def.
4. But $AB : A'B' = BC : B'C'$.	4. By hyp.
5. Then, since $AD = A'B'$, $\therefore DE = B'C'$.	5. § 328
6. $\therefore \triangle ADE \cong \triangle A'B'C'$.	6. Why?
7. $\therefore \triangle ABC \sim \triangle A'B'C'$.	7. Ax. 8

Ex. 1. In any triangle, the lines joining the end points of the three medians form a triangle similar to the given triangle.

Ex. 2. In triangle ABC, CD and AE are medians crossing at O.

Prove: $\triangle ODE \sim \triangle OCA$.



362. By §§ 356 and 361, when *either* of the conditions for similarity of polygons (§ 353) is true for two triangles, the other condition follows as a necessary consequence.

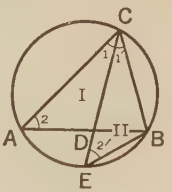
363. *Similar triangles are the usual means of proving that lines are proportional.* The steps are as follows:

1. *Select two triangles each having two of the given lines.*
2. *Prove that the two triangles are similar.*
3. *Derive the proportion by showing that the lines are corresponding sides of the triangles.* (§ 351, 2.)
4. *Then use alternation and inversion if necessary.*

In step 1, it will often be helpful to write the required proportion as a fractional equation and assume that the two lines in each fraction are corresponding sides in the two triangles. Then the lines in the two numerators will be sides in one of the triangles and the three distinct letters used to represent these lines will be the vertices of that triangle. Likewise for the lines in the two denominators.

These triangles may be represented as I and II, and the angles opposite the corresponding sides as 1 and 1′, 2 and 2′. It becomes evident at once what pairs of angles must be proved equal to make I similar to II. If the third pair of angles are identical, then it is necessary to prove only one other pair of angles equal.

It is sometimes necessary to draw lines between points to form the triangles indicated by the three letters which are to form its vertices.



Ex. ABC is an inscribed triangle and CE bisects angle ACB.

Prove: $\frac{AD}{EB} = \frac{CD}{CB}$.

Analysis:

1. Here AD and CD suggest $\triangle ADC$, that is, $\triangle$I, and EB and CB suggest $\triangle EBC$, that is, $\triangle$II.
2. Now $\triangle \text{I} \sim \triangle \text{II}$ if $\angle 1 = \angle 1'$ and $\angle 2 = \angle 2'$. (See drawing.)
3. Hence show these pairs of $\measuredangle$ equal.

Proof:

Left to the pupil.

364. *To prove that the product of two lines equals the product of two other lines,* use § 329, then proceed as in § 363 and take the product of the means and the extremes of the proportion derived in step 3.

365. *To prove that four lines are proportional when the lines are not found in two similar triangles,* find a third ratio equal to each of the two given ratios.

Exercises — Group 72

1. If ABC is a right triangle and CD is the altitude upon the hypotenuse, prove: (*a*) $\triangle ADC \sim \triangle ABC$; (*b*) $\triangle DBC \sim \triangle ABC$.

2. $ABCD$ is a parallelogram and P any point in BC. If AP and DC produced meet in O, prove: $\triangle ABP \sim \triangle OPC \sim \triangle OAD$.

3. In ex. 2, if DB cuts AO in E, prove: $EP : EA = EA : EO$.

SUG. See § 365. This third ratio is $EB : ED$.

4. ABC is an inscribed triangle and chord AD bisects angle A and cuts BC at P.

Prove: $AD : CD = AC : PC$.

5. The diagonals of a trapezoid divide each other proportionally.

6. Diameter RS is extended to D and a perpendicular at D meets chord NS produced at M.

Prove: $RS : SM = NS : SD$.

7. (*a*) ABC is an inscribed triangle, AD an altitude, and AE a diameter.

Prove: $AB : AD = AE : AC$.

(*b*) Suppose $\angle C$ is obtuse.

8. In ex. 7, prove: $AD : AC = BD : EC$.

9. If P is any point in chord AB and PC is perpendicular to diameter AD, prove: $AC \times AD = AB \times AP$.

10. The product of any side of a triangle and its altitude is equal to the product of any other side and its altitude.

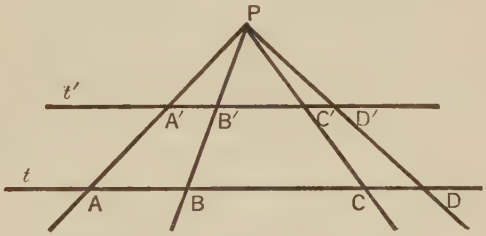
11. In the triangle ABC altitudes AD and BE meet in P.

Prove: $AP \times BD = BP \times AE$.

12. In ex. 11, prove: $BC \times EP = EC \times AP$.

Proposition VIII. Theorem

366. *If two parallels are cut by three or more transversals passing through a common point, the corresponding segments of the parallels are proportional.*



Given: Transversals PA, PB, PC, PD intersecting the parallels t and t' in A, B, C, D and A', B', C', D' respectively.

To prove: $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$.

Analysis:

Find a third ratio equal to each consecutive pair of ratios.

Proof:

Left to the pupil.

Discussion:

1. Where is the fact used that the transversals pass through a common point?
2. Name six pairs of similar triangles in the drawing.

Ex. 1. From the trisection points of side AB of triangle ABC lines are drawn to C. Prove that these lines trisect all lines parallel to AB and terminated by the other two sides.

Ex. 2. AB is a chord of circle O and radius OC is perpendicular to AB. If the tangent at C is drawn and the radii to A and B extended to meet this tangent at D and E respectively, prove that $DC = EC$.

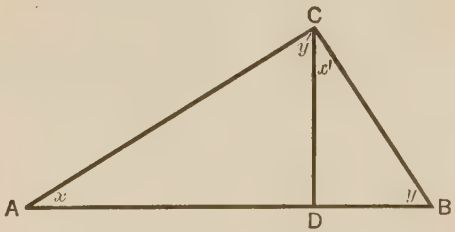
Proposition IX. Theorem

367. *If in a right triangle the perpendicular is drawn from the vertex of the right angle to the hypotenuse:*

I. *The two triangles thus formed are similar to the given triangle and to each other.*

II. *The perpendicular is the mean proportional between the segments of the hypotenuse.*

III. *Each leg of the given triangle is the mean proportional between the hypotenuse and the adjacent segment.*



Given: Rt. $\triangle ABC$, and CD the $\perp$ from C to the hypotenuse AB.

To prove: I. $\triangle ADC \sim \triangle ABC \sim \triangle DBC$.

II. $AD : DC = DC : DB$.

III. $AD : AC = AC : AB$ and $DB : BC = BC : AB$.

Analysis:

I. 1. Under what conditions are two rt. $\triangle$ similar?

2. Show $\triangle ADC \sim \triangle ABC$; $\triangle DBC \sim \triangle ABC$; then $\triangle ADC \sim \triangle DBC$.

II–III. See § 363.

Proof:

Left to the pupil.

368. COROLLARY. *If a perpendicular is let fall from any point in a circle upon a diameter, the perpendicular is the mean proportional between the segments of the diameter.*

* **Ex. 1.** In circle O, PA is a tangent and AB a diameter. If PB cuts the circle in C, prove: $(AC)^2 = BC \times CP$.

* **Ex. 2.** Using the figure for ex. 1, prove: $(AB)^2 = BP \times BC$; $(AP)^2 = PB \times PC$.

* **Ex. 3.** How is it shown in § 367 that $(DC)^2 = AD \times DB$? $(AC)^2 = AD \times AB$? $(BC)^2 = DB \times AB$?

Use the drawing for § 367 for ex. 4 to 7:

Ex. 4. If $AD = 9''$, $DB = 4''$, find DC, AC, and BC.

Ex. 5. If $CD = 8''$, $AD = 16''$, find DB, CB, and CA.

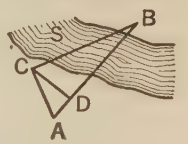
Ex. 6. If $AC = 6''$, $AD = 4''$, find AB and BC.

Ex. 7. If $BC = 8''$, $AD = 12''$, find DB, AC, and DC.

SUG. Let $x = DB$ and use § 367, III.

* **Ex. 8.** Two tangents are drawn to a circle at the ends of a diameter and a third tangent is intercepted between them. Prove that the radius is the mean proportional between the segments of the third tangent made by the point of tangency.

Ex. 9. To find the distance DB across a river S, one observer stands at D, another walks along the line DA, while a third walks to the left at right angles to AB for a convenient distance DC. The right angle BCA is observed and the observer walking along DA is signaled to stop. DC and DA are then measured.



(*a*) Show that $DB = (DC)^2 \div DA$.

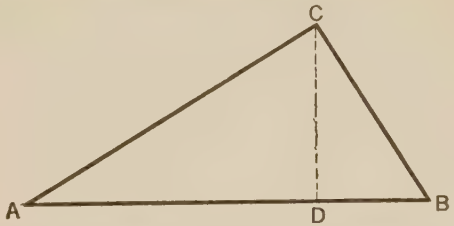
(*b*) If $DC = 75$ yd. and $DA = 25$ yd., find the distance DB.

Ex. 10. In a given circle AB is a diameter and F is any point in the semicircle AFB. If AF and BF produced meet the tangents at B and A in points C and D respectively, prove that AB is the mean proportional between AD and BC.

Ex. 11. If the sides of two triangles are $6''$, $8''$, $10''$ and $15''$, $20''$, $25''$ respectively, are the two triangles similar?

Proposition X. Theorem

369. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.*



Given: Rt. $\triangle ABC$, with AB the hypotenuse.

To prove: $(AB)^2 = (AC)^2 + (BC)^2$.

Analysis:

1. By § 367, III, to what is $(AC)^2$ equal? $(BC)^2$?
2. Hence show that $(AC)^2 + (BC)^2 = AB \times AB = (AB)^2$.

Proof:

Left to the pupil.

370. Corollary. *The difference of the square of the hypotenuse and the square of one leg equals the square of the other leg.*

371. § 369 refers to the numerical measures of the sides.

Ex. 1. If $AC = 20''$, $BC = 15''$, find AB.

Ex. 2. If $AB = 60''$, $AC = 45''$, find BC.

Ex. 3. If $BC = 8''$, $AB = 12''$, find AC.

Ex. 4. If $BC = 24''$, $BD = 14.4''$, find AB, AC, and DC.

Ex. 5. If $AC : BC = 3 : 4$ and $AB = 50''$, find AC and BC.

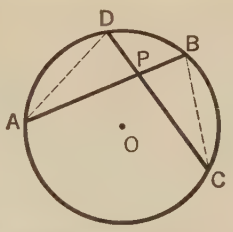
Ex. 6. If $AB : AC = 3 : 2$ and $BC = 35''$, find AB and AC.

372. A *constant* is a quantity which remains the same throughout a given problem or discussion.

Thus, the size of all angles inscribed in a given segment of a circle is constant, although the position of the vertex may change.

Proposition XI. Theorem

373. *If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.*



Given: Circle O and chords AB and CD intersecting in P.

To prove: $AP \times PB = CP \times PD$.

Analysis:

See § 364.

Proof:

Left to the pupil.

374. Corollary. *The product of the segments of any chord through a fixed point within a circle is constant.*

Draw the diameter through that point.

Discussion:

1. In stating that the product of segments of any chord through P is constant, does this mean that the *two segments* remain the same for all positions? Explain.
2. Would any other construction have given similar triangles from which the two products could be proved equal?

Ex. 1. In § 373, if $PB = 2''$, $AP = 9''$, $DP = 3''$, find CP.

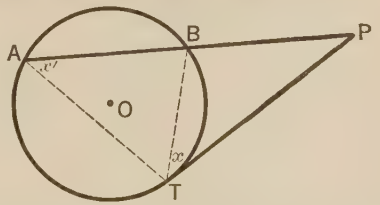
Ex. 2. If $AP : PB = 4 : 3$, $PD = 6''$, $CP = 8''$, find AP and PB.

Ex. 3. AB is a diameter, P a point on AB, and $AP = 3''$, $PB = 12''$. Find the length of the chord perpendicular to AB at P.

Ex. 4. Find the length of the shortest chord through a point $5''$ from the center of a circle whose radius is $13''$.

Proposition XII. Theorem

375. *If from a point without a circle a tangent and a secant are drawn to the circle, the tangent is the mean proportional between the secant and its external segment.*



Given: Tangent PT touching circle O at T, and secant PA cutting the circle in B.

To prove: $PA : PT = PT : PB$.

Analysis and Proof:

Left to the pupil.

376. Corollary 1. *The product of any secant from a fixed point without the circle and its external segment is constant.*

The length of a tangent from that point is constant.

377. Corollary 2. *If two or more secants are drawn to a circle from a fixed point without the circle, the product of one secant and its external segment is equal to the product of any other secant and its external segment.*

Discussion:

1. Is § 375 true when A coincides with B?
2. To what statement would § 375 reduce if TA were a diameter?
3. Could TB ever be a diameter? Prove your statement.
4. Is arc TB between O and P always a minor arc?

378. The *perimeter* of a polygon is the sum of the sides of the polygon.

Exercises — Group 73

Use the figure for § 375 in ex. 1 to 3:

1. If $PA = 18$ in. and $PB = 8$ in., find PT.

2. If $PT = 24$ in. and $PB = 16$ in., find AP.

3. If PBA passes through the center of the circle, find the radius of the circle if $PT = 8$ ft. and $PB = 4$ ft.

* **4.** Prove that tangents to two intersecting circles from any point in the common chord produced are equal.

5. Two circles are tangent internally and from any point in the common tangent secants are drawn to the two circles. Prove that the products of the secants and their external segments must be equal.

6. A balloon ascends along a line inclined 60° to the horizontal with a velocity of 20 mi. an hour. What is its altitude after 15 minutes of time? What is its velocity estimated in a horizontal direction?

7. A smokestack 75 ft. high is held by 3 wires attached 10 ft. from the top. What is the length of each wire if each is fastened in the ground 40 ft. from the stack?

8. At a certain hour of the day a flagpole casts a shadow of 20 ft. at the same time that a 10-foot pole casts a shadow of 6 ft. Find the height of the flagpole.

9. If one side of a square is 12″, find the diagonal.

10. If one side of an equilateral triangle is 16″, find the altitude.

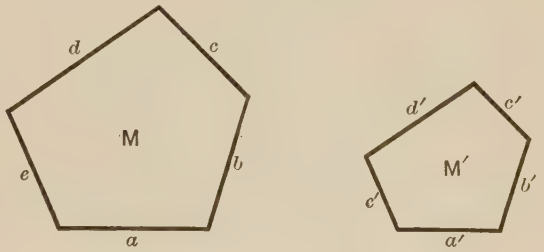
11. In a 60°–30° triangle if the shorter leg is 8″, find the hypotenuse and the other leg.

12. What position of the chords in § 373 would make the two segments of one chord equal? the two segments in each chord equal?

13. The common internal tangent to two circles divides the center segment into two parts which have the same ratio as the radii of the two circles.

Proposition XIII. Theorem

379. *The perimeters of two similar polygons are to each other as any two corresponding sides.*



Given: Two similar polygons M and M', with perimeters P and P', and any two corresponding sides a and a'.

To prove: $\frac{P}{P'} = \frac{a}{a'}$.

Analysis:

1. Express the required proportion in terms of the sides of the polygons.
2. What series of ratios are equal from the two polygons?
3. What property of proportion will now give the proportion as written in step 1?

Proof:

Left to the pupil.

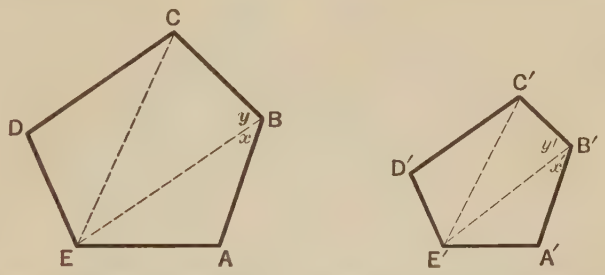
Ex. 1. Two similar polygons have perimeters in the ratio 3 : 5. Find the side in the second polygon corresponding to side 9 in. in the first.

Ex. 2. The sides of a polygon are 3, 5, 4, 7, 10 in. Find the sides of a similar polygon if the ratio between their perimeters is 3 : 4.

* **Ex. 3.** Prove that the perimeters of two similar polygons are in the same ratio as any two corresponding diagonals.

Proposition XIV. Theorem

380. *If two polygons are similar, they may be decomposed into the same number of triangles similar each to each and similarly placed.*



Given: Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$, each divided into triangles by diagonals drawn from corresponding vertices E and E'.

To prove: $\triangle EAB \sim \triangle E'A'B'$; $\triangle EBC \sim \triangle E'B'C'$ . . .

Analysis:

1. Show that $\triangle EAB \sim \triangle E'A'B'$, by § 359.
2. Show that $\triangle EBC \sim \triangle E'B'C'$, in the same way.
3. What $\measuredangle$ in these $\triangle$ may be proved equal by hyp. and step 1?
4. What proportion is necessary to make the $\triangle$ similar? (See § 365.)
5. Similarly prove the other triangles similar in order.

Proof:

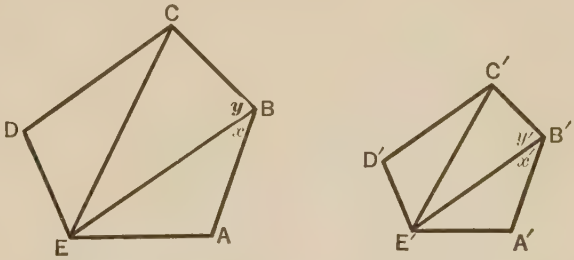
Left to the pupil.

Discussion:

1. Why is the number of $\triangle$ the same in each polygon?
2. Where is it shown that the $\triangle$ are similarly placed?
3. Could any vertex have been used instead of E? Must two corresponding vertices be used? Why?
4. Are both properties of similar polygons used? Explain.

Proposition XV. Theorem

381. *If two polygons are composed of the same number of triangles similar each to each and similarly placed, they are similar.* (Converse of Th. XIV.)



Given: Polygons $ABCDE$ and $A'B'C'D'E'$, with $\triangle EAB \sim \triangle E'A'B'$; $\triangle EBC \sim \triangle E'B'C'$; $\triangle ECD \sim \triangle E'C'D' \cdots$

To prove: Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

Analysis:

1. What pairs of $\measuredangle$ must be proved equal if the polygons are to be similar?
2. What pairs of $\measuredangle$ are known to be equal?
3. What series of ratios must be proved equal if the polygons are to be similar?
4. To what third ratio is each consecutive pair in this series equal?

Proof:

Left to the pupil.

Exercises — Group 74

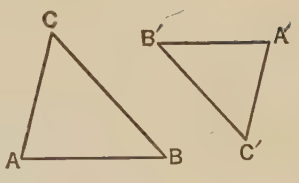
Similar Triangles

1. P is any point in arc APB and C is the mid-point of arc ACB. If chord AB cuts chord CP in E, prove that $\triangle APC \sim \triangle AEC$.

2. If P is any point in side AB of triangle ABC and F, K, E are the mid-points of AP, PB, and PC, prove that $\triangle FKE \sim \triangle ABC$.

3. Prove ex. 2 when P is any point (*a*) inside the triangle; (*b*) outside the triangle.

*** 4.** Two triangles whose sides are respectively parallel are similar.



1. Corr. $\measuredangle$ are $=$ or sup. Why?
2. Four possibilities arise:
   (*a*) Each pair of corr. $\measuredangle$ sup.
   (*b*) Two pairs of corr. $\measuredangle$ sup., one pair of corr. $\measuredangle$ equal.
   (*c*) One pair of corr. $\measuredangle$ sup., two pairs of corr. $\measuredangle$ equal.
   (*d*) Three pairs of corr. $\measuredangle$ equal.
3. Show (*a*) and (*b*) impossible, by § 152.
4. Hence the triangles are similar, by § 357.

*** 5.** Two triangles whose sides are respectively perpendicular are similar.

6. A triangle ABC is inscribed in a circle and at A a tangent is drawn. A chord parallel to the tangent is drawn cutting AB and AC. Prove that the triangle cut off is similar to triangle ABC.

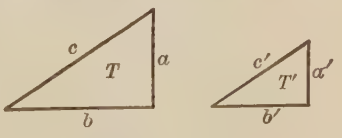
7. ABC and DEF are two isosceles triangles on bases BC and EF respectively. If $AB : BC = DE : EF$, show that the two triangles are similar.

8. Two circles intersect in P and Q and at P tangents are drawn to the two circles, cutting them in A and B respectively.

Prove: $\triangle APQ \sim \triangle BPQ$.

*** 9.** Two triangles are similar if two sides and the median to one of these sides in one triangle are proportional to the corresponding sides and median in the other.

*** 10.** If the hypotenuse and a leg of one right triangle are proportional to the hypotenuse and a leg of another, the two triangles are similar.



Given: $c : b = c' : b'$.

Prove: $\triangle T \sim \triangle T'$.

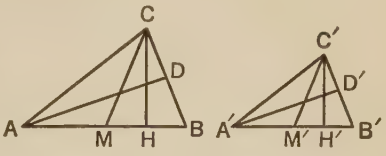
Sug. 1. Use § 360.

2. From the given proportion obtain the necessary proportion by the use of §§ 340, 337, and 370.

Exercises — Group 75

LINES PROPORTIONAL

Read §§ 363, 364, 365.



***1.** In two similar triangles any two corresponding sides are proportional to (*a*) any two corresponding altitudes; (*b*) any two corresponding medians; (*c*) the bisectors of any two corresponding angles.

Prove: $AC : A'C' = CH : C'H' = CM : C'M' = AD : A'D'$.

***2.** If one chord of a circle is bisected by another, either segment of the first is a mean proportional between the two segments of the other.

3. A triangle ABC is inscribed in a circle. M is the mid-point of arc AMC and BM intersects AC in D.

Prove: $AB : BC = AD : DC$.

4. AEB is a diameter and DEF any chord perpendicular to AB. Another chord AC cuts DF in K.

Prove: $AB \times AE = AC \times AK$.

5. D is the mid-point of side BC of triangle ABC and the bisectors of angles ADB and ADC meet AB and AC in E and F respectively.

Prove: EF is parallel to BC. (See § 345.)

6. In the isosceles triangle ABC ($CA = CB$), AF is the altitude on side BC.

Prove: $2\,BC \times BF = (AB)^2$.

SUG. Draw the bisector of $\angle C$.

7. From the vertex B of the inscribed isosceles triangle ABC a chord BE is drawn, cutting the base in D.

Prove: $(AB)^2 = BD \times BE$.

8. Two circles are tangent internally at P. PB and PD are chords of the larger circle cutting the smaller circle in A and C respectively.

Prove: $PA : PB = PC : PD$.

Exercises — Group 76

MISCELLANEOUS

* **1.** Prove that two parallelograms are similar if they have an angle equal and the included sides proportional.

2. Prove that the common chord of two intersecting circles bisects the common tangents of the two circles.

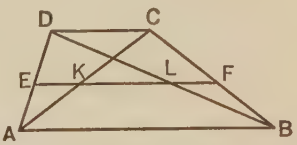
3. If two circles are tangent externally and through the point of contact a secant is drawn, the chords formed are proportional to the radii of the two circles.

4. In any triangle the median to one side bisects all lines parallel to that side and terminated by the other two sides. (See § 366.)

5. If the line of centers of two circles meets the circles in consecutive points A, B, C, D, and meets a common external tangent in P, show that $PA \times PD = PB \times PC$.

SUG. See ex. 7, p. 168; also § 365.

6. $ABCD$ is a trapezoid and EF is any line parallel to the bases and cutting AC in K and BD in L.



Prove: $EK = LF$.

SUG. Show that $EK : DC = LF : DC$.

7. D is the mid-point of side BC of triangle ABC and any line is drawn from C cutting AD in E and AB in F.

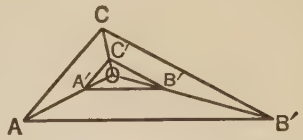
Prove: $\dfrac{AE}{ED} = 2 \times \dfrac{AF}{FB}$, *i.e.* $\dfrac{AE}{ED} = \dfrac{AF}{\frac{FB}{2}}$.

SUG. Draw $DK \parallel CF$.

8. Two triangles are similar if an angle of one equals an angle of the other and the corresponding altitudes from the other two vertices are proportional.

9. If two circles are tangent at T and through T three lines are drawn meeting the circles in P, Q, R and P', Q', R' respectively, prove that $\triangle PQR \sim \triangle P'Q'R'$. (See ex. 6, p. 168.)

10. O is any point inside the triangle ABC, B' is any point in OB, $A'B' \parallel AB$, $B'C' \parallel BC$, and A' and C' are joined.



Prove: $\triangle A'B'C' \sim \triangle ABC$.

Sug. Use §§ 359 and 365.

11. Prove ex. 10 when O is any point outside the triangle.

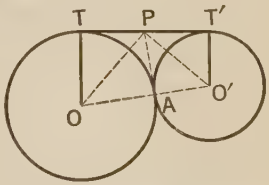
12. Prove that the line through O, the intersection of the diagonals of a trapezoid, parallel to the bases and terminated by the legs, is bisected by O. (See ex. 6.)

13. Tangents drawn at the vertices of an inscribed isosceles triangle form another isosceles triangle.

***14.** If two triangles are similar, the radii of their inscribed circles are in the same ratio as any two corresponding sides.

Sug. Draw radii to two corr. sides and use ex. 1 (*a*), p. 232.

15. If two circles are tangent externally, the common external tangent is the mean proportional between the diameters.



Prove: $2\,OT : TT' = TT' : 2\,O'T'$, or $OT : TP = PT' : O'T'$. Explain.

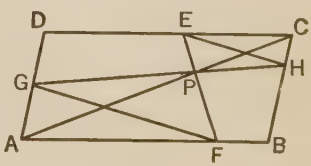
16. AB and AC are the equal sides of an isosceles triangle. The perpendicular bisector of AB meets BC in D.

Prove: $(AB)^2 = BC \times BD$.

Sug. Draw AD.

***17.** In the figure for Th. X (§ 369), show that if any two lines (not segments of the same line) are given, all the other lines are determined.

18. P is any point in the diagonal AC of parallelogram $ABCD$. EPF and GPH join points in opposite sides of the parallelogram and EH and GF both cut AC.



Prove: $\triangle PGF \sim \triangle PHE$.

Sug. Use §§ 359 and 365.

19. Two chords AB and CD intersect in M, A being the midpoint of arc CAD. Prove that the product $AB \times AM$ remains constant as the chord AB turns about the fixed point A.

Sug. Draw diameter AE; then draw BE.

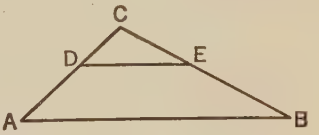
Exercises — Group 77

Numerical

1. Determine whether the two triangles in each of the following are similar:

(*a*) $a = 12''$, $b = 16''$, $c = 20''$; $a' = 6''$, $b' = 8''$, $c' = 10''$
(*b*) $a = 3\frac{1}{2}''$, $c = 4\frac{2}{3}''$, $B = 50°$; $a' = 2\frac{1}{2}''$, $c' = 3\frac{1}{3}''$, $B' = 50°$
(*c*) $b = 6''$, $c = 10''$, $m_c = 12''$; $b' = 4\frac{1}{2}''$, $c' = 7\frac{1}{2}''$, $m_c' = 9''$
(*d*) $b = 5''$, $c = 6''$, $h_c = 3''$; $b' = 3\frac{1}{3}''$, $c' = 4''$, $h_c' = 2''$

2. $AC = 10$, $BC = 15$, $AD = 6$, $CE = 6$. Is DE parallel to AB?



3. In the drawing of ex. 2, $AB = 24''$, $BC = 16''$, $AC = 18''$. Where must D be selected in AC so that line DE, parallel to AB, shall equal $9''$? What will be the segments of BC?

4. Two circles have radii $6''$ and $5''$ and their center segment is $22''$. Find the length (*a*) of their common internal tangent; (*b*) of their common external tangent.

5. How far from a circle whose radius is $6''$ must a point be selected so that the secant from that point through the center of the circle shall be twice the tangent from that point?

6. AB, a diameter of a circle, equals $12''$ and AC is a chord such that angle BAC equals 30°. Find the length of AC.

7. Find the diagonal of a square if one side is $18''$.

8. Find one side of a square if a diagonal is $24''$.

9. Find the diagonal of a rectangle if two adjacent sides are $15''$ and $20''$.

10. The diagonals of a rhombus are 12″ and 16″. Find one side.

11. One side of an equilateral triangle is 8″. Find the altitude.

12. The altitude of an equilateral triangle is 9″. Find one side.

13. The bases of a trapezoid are 12″ and 4″, the legs are 6″ and 8″. Find the sides of the triangles formed by producing the legs to meet.

14. At what point in a diameter of a circle whose radius is 5′ should a chord be drawn at right angles to the diameter if the length of the chord is to be 6′?

15. How far from a circle must a point be selected so that tangents to the circle from that point will form with each other an angle of (*a*) 60°? (*b*) 90°?

16. The sides of a triangle are 6″, 8″, and 12″. What are the segments of side 12″ made by the bisector of (*a*) the opposite interior angle? (*b*) the opposite exterior angle?

17. In the figure for Th. X (§ 369), if $AD = 51.2''$ and $BC = 48''$, find all the other lines of the drawing.

18. The bases of a trapezoid are 9″ and 6″, the altitude is 4″. How far from the longer base must a line be drawn parallel to the bases so that the part intercepted between the legs shall equal 8″?

(See Note in Group 63, page 195.)

19. The legs of an inscribed right triangle are 12″ and 16″. Find the length of the chord which bisects the right angle.

Sug. See §§ 348, 373.

20. The diagonals of a rhombus are 20″ and 30″. Find one side of the inscribed square.

Sug. See ex. 9, p. 197.

21. One diagonal of a rhombus is 12″ and one side of its inscribed square is 4″. Find one side of the rhombus.

22. The radii of two circles are 24″ and 8″. How far apart must their centers be if the common external tangent is 30″ long?

23. From a point 6″ from a circle whose radius is 8″ two secants are drawn, one passing through the center, the other intercepting a chord 1″ long. Find the external segment of the second secant.

Exercises — Group 78

APPLICATIONS

1. A rope fastened to the top of a flagpole is 3 ft. longer than the pole and when it is stretched out it reaches the ground 15 ft. from the foot of the pole. What is the height of the pole?

2. A ship sails along a coast 5 mi. away at the rate of 20 mi. an hour. A gun on shore has a range of 13 mi. How long is the ship in the range of the gun?

3. In physics it is shown that when two forces act together on a body at an angle the *resultant* is represented in direction and value by the diagonal of a parallelogram two of whose sides are represented by lines in the same ratio as the two forces.

A man swims at right angles to the bank of a stream at the rate of 3.5 mi. an hour. If the current flows 7.5 mi. an hour, at what rate is the man moving? in what direction?

4. Two forces of 80 lb. and 100 lb. act on a body at 60°. Find the direction in which and the force with which the body moves.

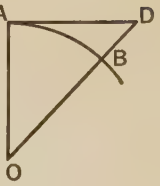
5. A teeter board 14 ft. long is supported at the center by a block 3 ft. high. How high can each end of the board rise? Would the height be the same for boards of different lengths?

6. A person 5 ft. tall is walking along a level street toward an arc light. At point A he observes that his shadow is 15 ft. long. He walks toward the light 20 ft. in a straight line to point B and observes that his shadow is now 10 ft. long. How high is the arc light and how far is A from a point directly under the light?

7. A tree situated on the bank of a stream subtends an angle of 60° at a point A on the opposite bank. At point B, 40 ft. directly back of A, the angle subtended by the tree is 45°. Find the width of the stream and the height of the tree.

8. An automobile goes 15 mi. along one straight road, turns 60° to the left on another straight road and goes 10 mi., then turns 90° to the left on another straight road and goes 6 mi. How far is it from the starting point?

9. A level line made by an instrument at any point on the earth is tangent to the surface. Hence the line of *true level* is below that of *apparent level*. In the figure AD represents the line of apparent level and AB the line of true level. Hence DB is the correction for the earth's curvature.

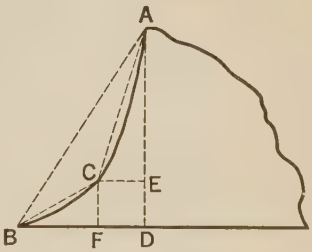


(*a*) Show that $(AD)^2 = DB(DB + 2\,BO)$.

(*b*) In this formula DB is so small compared with the diameter of the earth that it may be dropped from the parenthesis and hence $DB = (AD)^2 \div 2\,BO$. Explain.

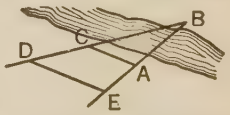
(*c*) Find at what distance from the shore a ship's mast 75 ft. above the water will disappear from sight, assuming that the radius of the earth is 3950 mi.

10. From a point B at the base of a mountain the angle DBA to the top is 60°. After walking 1 mi. toward the top up a plane making an angle of 30° to the horizontal to another point C, the angle BCA is observed to be 135°. Find the height (DA) of the mountain.



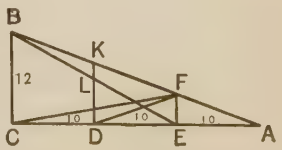
Sug. Find $CF = CE = FD$. Find BF; then BD, and hence AD.

11. To find the distance between two points A and B when one point A is accessible.



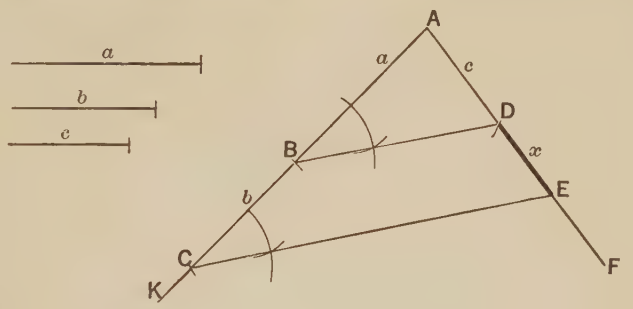
From A measure AC in any convenient direction. From a point D in line BC measure a line parallel to CA to a point E in line AB. Measure also AE. Then $AB = AC \times AE \div (DE - AC)$. Prove.

12. In the framework shown at the right CB, DK, and EF are each perpendicular to CA. From the given dimensions find the lengths of AB, DL, EL, CF.



Proposition XVI. Problem

382. *To construct a fourth proportional to three given lines.*



Given: Three lines a, b, and c.

Required: To construct a line x such that $a : b = c : x$.

Analysis:

Th. I, § 341, suggests a method, that is, get the four lines the segments of two sides of a $\triangle$ made by a line $\parallel$ the third side.

Construction and Proof:

Left to the pupil.

Discussion:

1. Can $\angle FAK$, used in the drawing, be of any size?
2. Does it make any difference in what order a, b, and c are taken? Make drawings to prove your answer.

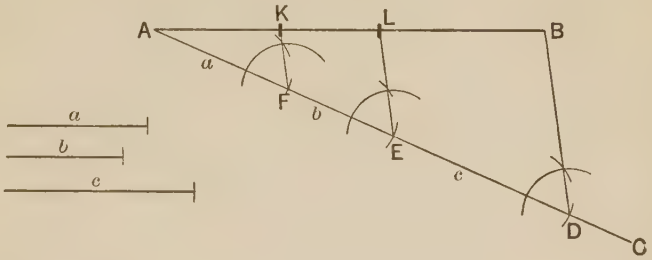
Ex. 1. Construct geometrically the fourth proportional to three lines $1\frac{1}{4}''$, $\frac{7}{8}''$, and $2\frac{1}{2}''$ long.

* **Ex. 2.** Given two lines a and b, construct the third proportional to a and b. May the lines be taken as a, b or b, a?

Ex. 3. If a, b, and c are three lines, construct a line x such that $a : x = b : c$.

Proposition XVII. Problem

383. *To divide a given line into parts proportional to any number of given lines.*



Given: Lines AB, a, b, and c.

Required: To divide AB into parts proportional to a, b, and c.

Analysis:

An extension of the method used in Prop. XVI is suggested.

Construction and Proof:

Left to the pupil. See § 344.

384. Corollary. *To divide a given line into parts proportional to two given lines.*

Discussion:

1. Does it make any difference in what order a, b, and c are taken?
2. Show that § 293 is a special case of § 383.

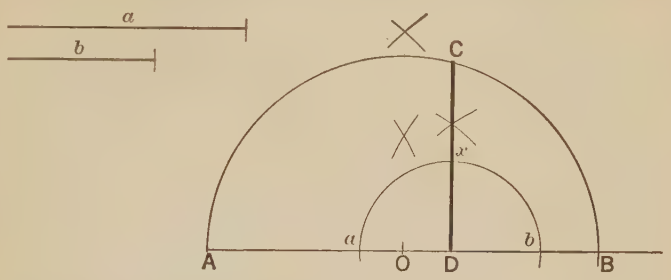
Ex. 1. Divide a line 3″ long into three parts in the ratio 1 : 2 : 5. (See § 386.)

Ex. 2. Divide a line 4″ long into two parts in the ratio $a : b$ where a and b are any two given lines.

Ex. 3. Find a point in a line $3\frac{1}{2}$″ long which divides the line into parts which have a given ratio.

Proposition XVIII. Problem

385. *To construct the mean proportional between two given lines.*



Given: Two lines a and b.

Required: To construct a line x such that $a : x = x : b$.

Analysis:

1. See § 368. To what lines would a, b, and x correspond?
2. Hence, what is the diameter of the circle? Where is the $\perp$ erected?

Construction and Proof:

Left to the pupil.

386. In representing a ratio expressed as a *numerical* fraction, any two lines may be used which have this ratio.

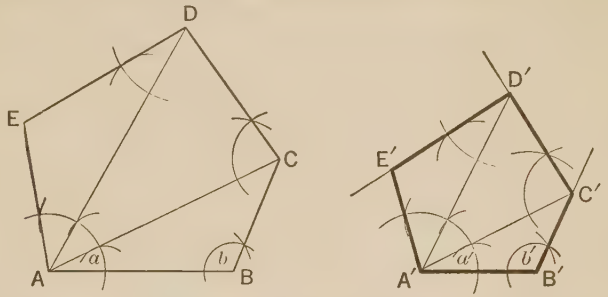
Thus, in representing the ratio 2 : 3, any line may be selected as a *unit*. Then the first term of the ratio would be represented by a line twice the unit, and the second term by a line three times the unit.

In representing a ratio expressed as a *literal* fraction, such as $a : b$, where the numerical value of the ratio is not given, we may consider the two terms a and b as any two given lines.

If a problem states that the ratio between two lines is given, but does not specifically mention what this ratio is, we may represent it as $m : n$, where m and n are to be treated as *given* lines.

Proposition XIX. Problem

387. *Upon a given line corresponding to one side of a given polygon to construct a polygon similar to the given polygon.*



Given: Line $A'B'$ corresponding to side AB of polygon $ABCDE$.

Required: To construct on $A'B'$ a polygon similar to polygon $ABCDE$.

Analysis:

1. By § 381, what condition makes two polygons similar?
2. Construct the required polygon to satisfy this condition.
3. What △ would be constructed first and what would be the method? (See § 357.)

Construction and Proof:

Left to the pupil.

Ex. 1. Construct geometrically the mean proportional between two lines $1\frac{1}{2}''$ and $3\frac{3}{8}''$ long. Check by determining the length of the required line algebraically.

* **Ex. 2.** Construct a line equal to $\sqrt{10}$, that is, $\sqrt{5 \times 2}$; to $\sqrt{6}$; to $2\sqrt{3}$; to $a\sqrt{2}$, that is, $\sqrt{2\,a^2} = \sqrt{2\,a \times a}$, where a is a given line.

Ex. 3. Prove geometrically that $\sqrt{12} = 2\sqrt{3}$.

Ex. 4. Given a pentagon, construct a similar pentagon with sides one half as long; $\sqrt{2}$ times as long. (See § 386 and ex. 2.)

Exercises — Group 79

Read § 316.

Given three lines a, b, and c:

* **1.** Construct $x = \frac{a \cdot b}{c}$; $x = \frac{a \cdot c}{2\,b}$; $x = \frac{b^2}{c}$; $x = \frac{a(b + c)}{c}$. In each case write the corresponding proportion, making x the last term, and use § 382.

* **2.** Construct $x = \sqrt{a^2 + b^2}$. (Use § 369.) Construct $x = \sqrt{a^2 - b^2}$ (if $a > b$).

* **3.** Construct $x = \sqrt{2\,ab}$. Show geometrically why, in this construction, segments of the same length are obtained whether we consider $x = \sqrt{(2\,a)b}$ or $x = \sqrt{(2\,b)a}$.

4. Divide a given line a into parts in the ratio $1 : \sqrt{2}$; $\sqrt{2} : \sqrt{3}$. (See § 383 and § 387, ex. 2.)

* **5.** Construct two lines, given their sum and their ratio. (See § 384.)

6. Construct two lines, given their difference and their ratio. (See §§ 337 and 386.)

7. Construct the triangle ABC, given a, b, and the ratio $b : c$.

8. Construct a triangle similar to a given triangle and having a given perimeter. (See ex. 33, p. 194.)

9. Construct a triangle similar to a given triangle and having a given altitude. (See ex. 3, p. 193.)

* **10.** Inscribe in a given circle a triangle similar to a given triangle.

* **11.** Circumscribe about a given circle a triangle similar to a given triangle.

12. In the prolongation of side AB of triangle ABC, find a point P such that $AP \times BP = (CP)^2$. (See § 375.)

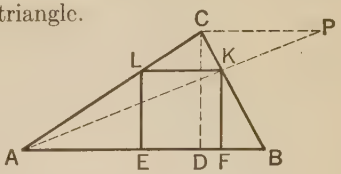
* **13.** Through point P in the minor arc of chord AB draw a chord that shall be bisected by AB. (See ex. 5, p. 168.)

14. Given an obtuse triangle. Draw a line from the vertex of the obtuse angle to the opposite side which shall be the mean proportional between the segments of that side. (See ex. 13; also ex. 2, p. 232.)

15. If R is any point in the side AB of triangle ABC, find a point T in the prolongation of side AC such that RT is bisected by BC.

SUG. Through R draw a $\parallel$ to BC cutting AC in R'.

16. Inscribe a square in a given triangle.

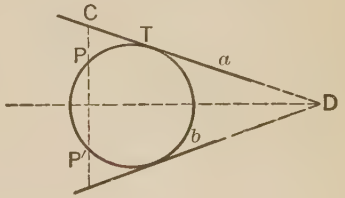


SUG. 1. $\frac{LK}{CP} = \frac{AL}{AC} = \frac{LE}{CD}$. Why?

2. $LK = LE$, $CP = CD$. Why?

3. Make $CP = CD$ and find K.

17. Construct a circle passing through a given point P and touching two given lines a and b.



SUG. 1. CP and CP' are known. Why?

2. Also $(CT)^2 = CP' \times CP$. Why?

3. Hence locate T by § 385.

18. In ex. 17 assume that a is parallel to b.

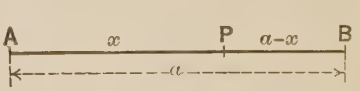
19. Construct a circle that shall pass through two given points A and B and touch a given line m. Consider the case where (*a*) AB is parallel to m (see § 305); (*b*) AB is not parallel to m (see § 375).

Sometimes it is helpful to consider the length of some line as the unknown (x) and to express the conditions of the problem, in terms of x, in the form of an equation. Then, from the solution of the equation, determine x geometrically. (See § 323.)

ILLUSTRATION 1. Find a point P in a given line AB, such that $(AP)^2 = AB \times PB$.

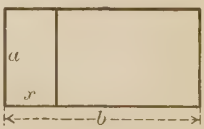
$$x^2 = a(a - x) = a^2 - ax.$$

$$x^2 + ax + \left(\frac{a}{2}\right)^2 = a^2 + \left(\frac{a}{2}\right)^2.$$



$$x = \frac{a\sqrt{5} - a}{2}.$$ Construct $a\sqrt{5}$; then $\frac{a\sqrt{5} - a}{2}$.

ILLUSTRATION 2. From a given rectangle cut off a similar rectangle by a line parallel to one side.



In the diagram, $b : a = a : x$. Construct x.

20. Find a point P in a given line AB such that $(AP)^2 = 2(BP)^2$.

21. Construct the triangle ABC, given b and the segments of c made by t_c.

22. In a given circle inscribe a rectangle, having given the ratio of the two sides.

Sug. Divide a diameter into the given ratio and use § 348.

23. Find P in the arc having a given chord AB so that $PA : PB$ shall equal a given ratio. (See § 348.)

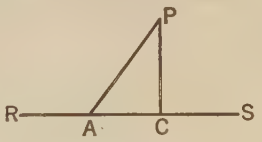
24. Divide a given trapezoid into two similar trapezoids by a line parallel to the bases.

25. Inscribe a square in a given semicircle.

Exercises — Group 80

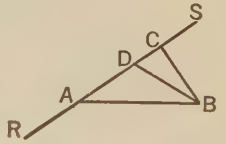
Applications

1. To construct with a 100-foot tape a line at right angles to RS at one of its points C.

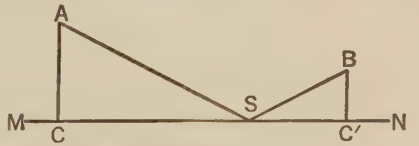


Locate A on RS 30 ft. from C. Hold the zero end of the tape at A, the 100-foot end at C, and bring together at P the 50-foot and 60-foot marks. The tape stretched tight forms a right angle at C. Explain.

2. Two houses A and B are 100 rd. apart. A is located on a straight road RS and B is 60 rd. from the road. Locate D on the the road the same distance from the two houses.



3. A and B are two camps situated 300 yd. and 150 yd. respectively from a straight road MN. Locate a common supply store S for the two camps on the road MN to the most advantage with respect to distance, the distance CC' being 900 yd.



388. Projection. The *projection of a point upon a line* is the foot of the perpendicular from the point to the line.

The *projection of one line upon another* is the locus of the projection of all its points. It is therefore the line between the projections of the extremities of the first line upon the second, produced if necessary.

Fig. 1

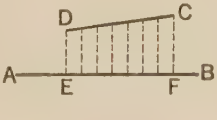


Fig. 2

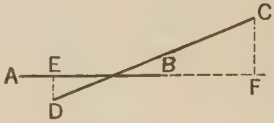


Fig. 3

Thus, F (Fig. 1) is the projection of point P upon line AB and EF (Fig. 2, 3) is the projection of line DC upon line AB.

Ex. 1. Is the projection of a point always a point? Explain.

Ex. 2. Is the projection of a line always a line? Explain.

Ex. 3. Can the projection of a line upon another line ever be greater than the line? less than the line? equal to the line? Explain.

Ex. 4. What is the projection of the hypotenuse upon one leg of a right triangle? the projection of one leg upon the other? the sum of the projections of the two legs upon the hypotenuse?

Ex. 5. To what is the sum of the projections of any two sides of a triangle upon the third side equal? Illustrate with acute, right, and obtuse triangles.

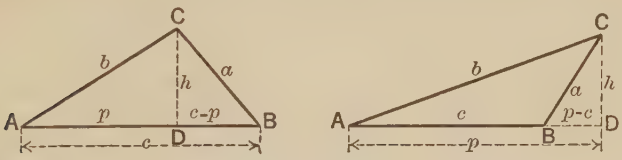
Ex. 6. To what is the sum of the projections of the four sides of any quadrilateral upon a diagonal equal?

Exercises — Group 81

Some Numerical Properties of Triangles

***1.** In any triangle the square of the side opposite an *acute* angle is equal to the sum of the squares of the other two sides *diminished* by twice the product of one of those sides and the projection of the other side upon it.

Given: $\triangle ABC$, $\angle A$ acute, and p the projection of b upon c.
Prove: $a^2 = b^2 + c^2 - 2pc$.



Analysis:

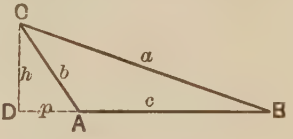
1. In either figure to what is a^2 equal, by § 369? Expand.
2. In this expression what parts occur that are not in the desired expression?
3. By what substitution may the desired form be obtained?

* **2.** In any obtuse triangle the square of the side opposite the *obtuse* angle is equal to the sum of the squares of the other two sides *increased* by twice the product of one of those sides and the projection of the other side upon it.

Given: $\triangle ABC$, $\angle A$ obtuse, and p the projection of b upon c.

Prove: $a^2 = b^2 + c^2 + 2pc$.

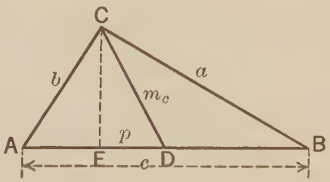
Sug. Use the method of ex. 1.



3. In any triangle prove: $p_c{}^b = \dfrac{b^2 + c^2 - a^2}{2c}$ where $p_c{}^b$ is the projection of b upon c and angle A is acute. Similarly for the other sides.

* **4.** In any triangle the sum of the squares of any two sides is equal to twice the square of one half the third side increased by twice the square of the median to that side.

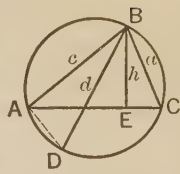
Prove: $a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2m_c{}^2$.



Analysis:

1. To what is a^2 equal in $\triangle DBC$?
2. To what is b^2 equal in $\triangle ADC$?

* **5.** From ex. 4, show that $m_c = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$.

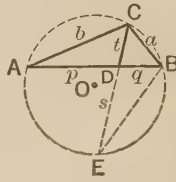


* **6.** In any triangle the product of two sides is equal to the product of the altitude on the third side and the diameter of the circumscribed circle.

Prove: $a \times c = h \times d$.

* **7.** In any triangle the product of two sides is equal to the square of the bisector of the included angle increased by the product of the segments of the third side made by this bisector.

Prove: $a \times b = t^2 + p \times q$.



Analysis:

1. $p \times q$ suggests § 373.
2. What ▲ involving a and b are similar?
3. Form a proportion involving a and b, and obtain an expression for $a \times b$.
4. Transform this to the desired form by § 373.

8. To what statement does ex. 7 reduce when $a = b$?

9. Prove: $t_c = \dfrac{2}{a+b}\sqrt{abs(s-c)}$ where $s = \frac{1}{2}(a+b+c)$.

SUG. See ex. 7 and § 348.

10. The sides of a triangle are 9″, 41″, and 40″. Find the median on side 41″.

11. Find the altitude upon side 30″ in a triangle whose sides are 8″, 24″, and 30″.

12. In ex. 11 find the length of the projection of side 8″ upon side 24″; of side 24″ upon side 30″.

Ex. 1 and 2 and Th. X furnish a method for determining whether a triangle is acute, right, or obtuse when the sides are given. If c is the longest side, $\angle C$ is the largest angle. Hence:

When $c^2 < a^2 + b^2$, the triangle is acute, by ex. 1.
When $c^2 = a^2 + b^2$, the triangle is right, by Th. X.
When $c^2 > a^2 + b^2$, the triangle is obtuse, by ex. 2.

13. Which of the following triangles are right? acute? obtuse?

(*a*) 5, 12, 13 (*c*) 42, 11, 44 (*e*) 30, 31, 16 (*g*) 32, 8, 24
(*b*) 16, 6, 17 (*d*) 24, 7, 25 (*f*) .8, 3.9, 3.6 (*h*) 1.5, .8, 1.6
(*i*) 8.1, 36, 36.9 (*j*) 140, 782, 818

BOOK FOUR

SURFACE MEASUREMENT

AREAS

389. *Any plane surface may be measured* by selecting some unit of surface and then determining the number of times this unit is contained in the given surface. (See § 264.)

390. The *unit of surface* most frequently used is a square whose side is of unit length. If this unit length is 1 inch, the unit of surface is a square whose side is an inch, called 1 square inch. If the unit of length is 1 foot, 1 yard, etc., the unit of surface is 1 square foot, 1 square yard, etc.

391. The *area of any plane surface* is the number that indicates how many square units of a given kind it contains.

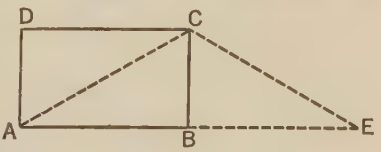
Thus, if the rectangle $ABCD$ contains the square U 12 times, then the area of the rectangle, in terms of U, is 12. That is, rectangle $ABCD = 12\ U$.

392. *Two plane figures are equal* if they have the same area, irrespective of their shape.

Some texts use the word *equivalent* in the same sense as the word *equal* is here used.

It follows from the definition (§ 65) that congruent figures are equal. Equal figures, however, are not necessarily congruent.

Thus, if $ABCD$ is a rectangle and if triangle ADC is transferred to the position BEC, then triangles ADC and ABC are congruent; also triangle AEC and rectangle $ABCD$ are equal, but not congruent.

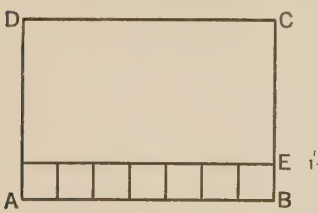


***Ex. 1.** Find the area of the rectangle whose base is 5″ and whose altitude is 3″.

Since 1″ is a common unit of measure of AB and AD, a square 1″ on a side, that is, 1 sq. in., may be used as the unit of measure. In one row, AE, there are then 5 squares, and since from A to D there are 3 rows, rectangle AC contains 15 squares, that is, the area of the rectangle $AC = 15U = 15$ (sq. in.) $= (3 \times 5)$ sq. in.

***Ex. 2.** Find the area of a rectangle whose base is $3\frac{1}{2}$″ and whose altitude is $2\frac{1}{2}$″.

Since $\frac{1}{2}$″ is a common unit of measure of AB and AD, a square $\frac{1}{2}$″ on a side, that is, $\frac{1}{4}$ sq. in., may be used as the unit of measure. In one row, AE, there are then 7 squares, and since from A to D there are 5 rows, rectangle AC contains 35 squares, that is, the area of rectangle $AC = 35U' = \frac{35}{4}$ (sq. in.) $= (2\frac{1}{2} \times 3\frac{1}{2})$ sq. in.

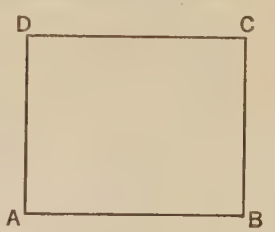


The same is true of any rectangle whose base and altitude have a common unit of measure. Such a common unit of measure can always be found when the numerical measure of the base and the altitude in terms of the same unit can be exactly expressed as an integer or as the quotient of one integer divided by another.

***Ex. 3.** Find the area of the rectangle whose base is $\sqrt{3}$ in. and whose altitude is $\sqrt{2}$ in.

AB and AD have no common unit of measure, hence no unit can be used, as in ex. 1 and 2, which will give a *whole number* of squares in the row along AB or a *whole number* of rows along AD. Since, however, the part left over on each line, when any unit is used, is less than the unit length (why?), it is evident that the smaller

the unit square used, the more nearly does each row contain a whole number of squares and AD a whole number of rows and, consequently, the more nearly does the product of these numbers equal the area of the rectangle AC. This is shown in the following table where unit squares whose areas are respectively .01 sq. in., .0001 sq. in., and .000001 sq. in. are used.



Side of U	$\sqrt{2}$	$\sqrt{3}$	Area
.1	1.4 ···	1.7 ···	2.38 ···
.01	1.41 ···	1.73 ···	2.4393 ···
.001	1.414 ···	1.732 ···	2.449048 ···

That is, the smaller the unit square, the nearer do the successive areas come to the product of the base and the altitude of the given rectangle, for $\sqrt{2} \times \sqrt{3} = \sqrt{6} = 2.4494897 \cdots$

If the sides of the rectangle have a common unit of measure, as in ex. 1 and 2, then the *number* of units of area is equal to the product of the *number* of units in the base by the *number* of units in the altitude when each is expressed in terms of one side of the unit square.

If the sides of the rectangle have no common unit of measure, as in ex. 3, it is evident that by making the side of the unit square successively smaller, the successive numbers of units of area grow constantly closer to the product of the *number* of units in the base by the *number* of units in the altitude.

A **constant** (see § 372) is a quantity which remains the same throughout a given problem or discussion.

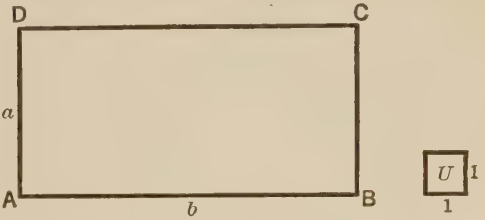
A **variable** is a quantity which continually changes throughout a given problem or discussion.

If a variable x approaches a constant a so that the difference between a and x becomes less and remains less than any assignable quantity, then a is called the **limit** of x.

Thus, in ex. 3 the *constant* area AC is the *limit* toward which the successive *variable* areas are tending as the side of the unit square is made smaller and smaller.

Proposition I. Theorem

393. *The area of a rectangle is equal to the product of its base and its altitude.*



Given: Rectangle AC, with base AB denoted by b, altitude AD denoted by a, and area denoted by R.

To prove: $R = b \times a$.

Case I. When AB and AD are commensurable.

Analysis:

Select a unit common to AB and AD and proceed as in ex. 1 and 2, p. 250.

Proof:

Left to the pupil.

Case II. When AB and AD are incommensurable.

Analysis:

The truth of this case may be found by a course of reasoning similar to that in ex. 3, p. 251.

394. Corollary 1. *Two rectangles are to each other as the products of their bases and their altitudes.*

Let R, R', b, b', and a, a' represent the two rectangles, the two bases, and the two altitudes respectively.

To what is R equal? R'? Hence what does $R : R'$ equal?

395. Corollary 2. *Two rectangles having equal bases are to each other as their altitudes.*

396. It should constantly be kept in mind that the product of two lines means the products of their numerical measures. Th. I is merely an abbreviated form of the statement: *The number of surface units in a rectangle is equal to the product of the number of linear units in the base by the number of linear units of the same kind in the altitude.*

397. The base and the altitude of a rectangle are often called the *dimensions* of the rectangle.

Since the area of a rectangle whose sides are a and b is $a \times b$, the product of two lines m and n, that is, $m \times n$, is often spoken of as "the rectangle of m and n," and means the rectangle whose dimensions are m and n.

In propositions relating to areas the words "rectangle," "triangle," etc. are frequently used for "area of rectangle," "area of triangle," etc.

Ex. 1. Find the area of a rectangle whose base and altitude are: (*a*) 5 ft. and $2\frac{1}{2}$ ft.; (*b*) 8 in. and $1\frac{1}{2}$ ft.

Ex. 2. The area of a rectangle is 35.88 sq. yd. The altitude is 5.2 yd. What is the base?

Ex. 3. The base and the altitude of a rectangle are in the ratio 3 : 2 and the area is 294 sq. in. Find the dimensions of the rectangle.

Ex. 4. Find the ratio of two rectangles 8′ by 12′ and 6′ by 9′.

Ex. 5. Two rectangles are in the ratio 2 : 3. If $a = 6''$, $b = 8''$, $b' = 12''$, find a'.

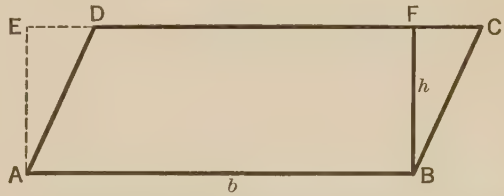
Ex. 6. The dimensions of a rectangle are 9″ and 12″. What are the dimensions of a rectangle $\frac{2}{3}$ as large cut off by a line (*a*) parallel to the base? (*b*) parallel to the altitude?

398. Any side of a parallelogram may be considered as its *base*. For a given parallelogram it is customary to consider the side upon which the parallelogram rests as the base.

When speaking of the base and the altitude of a parallelogram, a base and the corresponding altitude are always meant. Likewise for the base and the altitude of a triangle.

Proposition II. Theorem

399. *The area of a parallelogram is equal to the product of its base and its altitude.*



Given: $\square ABCD$, with base AB denoted by b and altitude BF denoted by h.

To prove: Area of $\square ABCD = b \times h$.

Analysis:

1. What figure would have $AB \times BF$ for its area? (§ 393.)
2. How may such a figure be constructed in the drawing?
3. Then what two figures in the drawing must be proved equal?
4. How is each obtained from the figure $ABCE$?
5. Then what must be proved to get the two figures equal?

Proof:

Left to the pupil.

400. Corollary 1. *Any two parallelograms are to each other as the products of their bases and their altitudes.*

401. Corollary 2. (*a*) *Two parallelograms having equal bases are to each other as their altitudes.*

(*b*) *Two parallelograms having equal altitudes are to each other as their bases.*

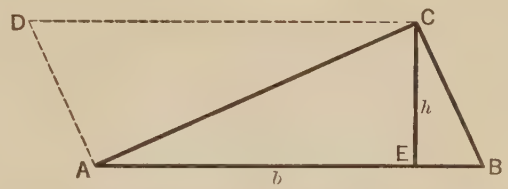
402. Corollary 3. *Two parallelograms having equal bases and equal altitudes are equal.*

Discussion:

Make an appropriate drawing, considering AD as the base.

Proposition III. Theorem

403. *The area of a triangle is equal to one half the product of its base and its altitude.*



Given: $\triangle ABC$, with base AB denoted by b and altitude CE denoted by h.

To prove: Area of $\triangle ABC = \frac{1}{2}\, b \times h$.

Analysis:

1. What area is represented by $b \times h$, by § 399?
2. Construct such a figure in the drawing and show that $\triangle ABC$ is one half its area.

Proof:

Left to the pupil.

404. Corollary 1. *Any two triangles are to each other as the products of their bases and their altitudes.*

405. Corollary 2. (*a*) *Two triangles having equal bases are to each other as their altitudes.*

(*b*) *Two triangles having equal altitudes are to each other as their bases.*

406. Corollary 3. *Two triangles having equal bases and equal altitudes are equal.*

407. Corollary 4. *A triangle is equal to one half a parallelogram having the same base and altitude.*

Discussion:

1. Make an appropriate drawing for Th. III, considering AC as the base.
2. Can this proposition be proved by using Th. I instead of Th. II? Explain.
3. Why is h the same for the given $\triangle$ and the $\square$?
4. To what statement does this theorem reduce if $\angle B$ = a rt. $\angle$?
5. Why is it undesirable to use Th. III as the fundamental proposition for areas, instead of Th. I?
6. Can the area of a $\triangle$ be independent of the size of its base $\measuredangle$?

Ex. 1. Find the area of a triangle whose base is 15 in. and whose altitude is 6 in.

Ex. 2. The area of a triangle is 176.86 sq. ft. The base is 24.2 ft. Find the altitude.

Ex. 3. In § 399, could the altitude have been drawn from *any* point in AB or CD? Why? What would have been the construction if the altitude had been drawn from D?

Ex. 4. Could Th. II, p. 254, be proved by using $ABFD$ instead of $ABCE$, as suggested in step 4? Would the same axiom be used?

Ex. 5. May the area of a $\square$ be independent of the size of its $\measuredangle$? Illustrate.

Ex. 6. The base of a parallelogram is 18 in. and the altitude is 6.4 in. Find the area.

Ex. 7. The area of a parallelogram is 112.48 sq. ft. and the base is 15.2 ft. Find the altitude.

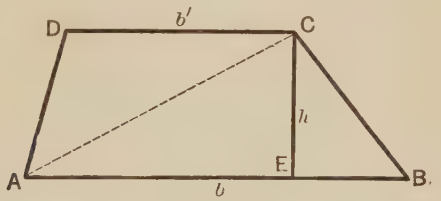
Ex. 8. Two sides of a parallelogram are 6″ and 10″ and the included angle is 30°. Find the area.

Ex. 9. The ratio between the base and the altitude of a triangle is 2 : 3. If the area of the triangle is 108 sq. in., find the base and the altitude.

Ex. 10. Two adjacent sides of a parallelogram are in the ratio 4 : 5 and the included angle is 60°. If the area of the parallelogram is $1280\sqrt{3}$ sq. ft., find the sides.

Proposition IV. Theorem

408. *The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.*



Given: Trapezoid $ABCD$, with bases b and b' and altitude h.

To prove: Area of trapezoid $ABCD = \frac{1}{2}h(b + b')$.

Analysis:

1. Draw a diagonal and make the proof depend upon Th. III.
2. What is the altitude of $\triangle ADC$? Why?

Proof:

Left to the pupil.

409. Corollary. *The area of a trapezoid is equal to the product of its altitude and its median.*

See ex. 8, p. 104.

Discussion:

1. To what does this theorem reduce if $b' = 0$? if $b = b'$?
2. Could this method be used to prove Th. III? Explain.

Ex. 1. The bases of a trapezoid are 5 ft. 4 in. and 3 ft. 10 in., and its altitude is 2 ft. 4 in. Find the area.

Ex. 2. The area of a trapezoid is 55 sq. in. The bases are 14 in. and 8 in. Find the altitude.

Ex. 3. The area of a trapezoid is 276 sq. in. The altitude is 12 in. and one base is 14 in. Find the other base.

Exercises — Group 82

* **1.** The area of a right triangle is equal to one half the product of the two legs.

2. Prove that the area of any right isosceles triangle equals one fourth the square on the hypotenuse of the triangle.

3. Find the area of a square whose side is 12 in.

4. If the diagonals of a quadrilateral are perpendicular to each other, the area of the quadrilateral equals one half the product of the two diagonals.

5. Prove that a median of a triangle divides the triangle into two equal parts. Are the parts necessarily congruent?

* **6.** The two diagonals of a parallelogram divide the parallelogram into four equal parts. Which parts are congruent?

7. $ABCD$ is a parallelogram and P is any point in CD or in its prolongation.

Prove: $\triangle PAB = \frac{1}{2}\,\square ABCD$.

8. In ex. 7, if the diagonals intersect in O, prove: $\triangle PAB = 2 \times \triangle OAB$.

9. $ABCD$ is a parallelogram and P is any point in diagonal AC or in its prolongation.

Prove: $\triangle ABP = \triangle ADP$.

10. The lines joining the mid-point of one diagonal of a quadrilateral to the opposite vertices divide the quadrilateral into two equal parts.

11. Prove Th. IV, using the following drawings:

12. The area of a rhombus is 216 sq. in. and one diagonal is 18 in. Find the other diagonal. (See ex. 12, p. 99.)

13. The base of a triangle is 18.6 in. and the altitude is 12.4 in. Find a side of the square whose area equals that of the triangle.

14. What must be the altitude of the rectangle whose base is 16 in. if it is equal to the triangle in ex. 13?

15. One side of an equilateral triangle is 24 in. Find the altitude.

16. Find the area of the triangle in ex. 15.

17. If the area of an equilateral triangle is 48 sq. in., find one side and the altitude.

SUG. Let $2a$ = one side and express the altitude and then the area in terms of a. Then solve for a.

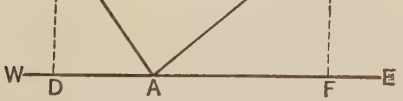
18. Find the area of a rhombus if the diagonals are 12 in. and 16 in.

19. Find the altitude of the rhombus in ex. 18.

20. The base of a triangle is 24 in. and the altitude is 18 in. Find the area of the triangle cut off by a line parallel to the base a distance of 15 in. from the base.

21. How much leather is needed to cover a window seat whose top is half a regular hexagon if the length of each shorter side is $4\frac{1}{2}$ ft.?

22. To find the area of a triangular tract of land the following measurements were made, EW being an east-west line:

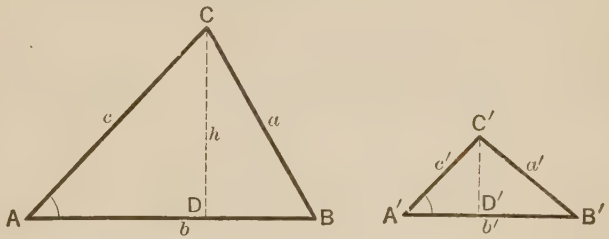


DB = 16 rd., DA = 12 rd., AF = 20 rd., FC = 18 rd. Find the area of triangle ABC, DB and FC being perpendicular to WE.

23. Find the area of the following plot of ground: Start at A, go east 18 ft. to B, north 24 ft. to C, northeast 16 ft. to D, north 12 ft. to E, west to F, directly north of A, and then south to A.

Proposition V. Theorem

410. *If two triangles have an angle of one equal to an angle of the other, their areas are to each other as the products of the sides including the equal angles.*



Given: $\triangle ABC$ and $A'B'C'$, with $\angle A = \angle A'$.

To prove: $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{b \times c}{b' \times c'}$.

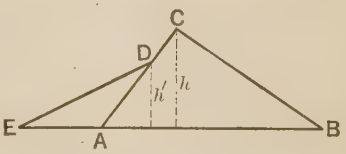
Analysis:

1. To what is the ratio $\dfrac{\triangle ABC}{\triangle A'B'C'}$ equal, by § 404? Express this as the product of two ratios.
2. To make this product equal to the required product what pairs of ratios must be shown equal? (See § 363.)

Proof:

Left to the pupil.

* **Ex. 1.** If two triangles have an angle of one supplementary to an angle of the other, their areas are to each other as the products of the sides including these angles.

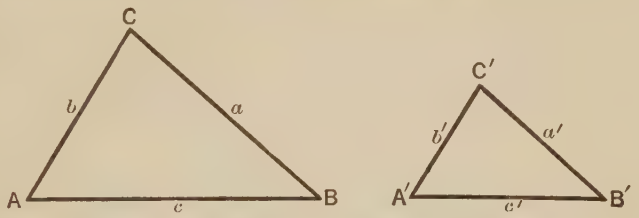


Sug. Place the two $\triangle$ as in the diagram at the right and use an outline similar to the one above.

Ex. 2. If two triangles have two sides of one respectively equal to two sides of the other and the included angles supplementary, the triangles are equal.

Proposition VI. Theorem

411. *The areas of two similar triangles are to each other as the squares of any two corresponding sides.*



Given: Similar $\triangle ABC$ and $A'B'C'$, with corresponding sides c and c'.

To prove: $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{c^2}{c'^2}$.

Analysis:

1. To what is the ratio $\dfrac{\triangle ABC}{\triangle A'B'C'}$ equal, by § 410? Express this as the product of two ratios.
2. How must these two ratios compare if their product is to equal the required ratio? Is this relation true? Why?

Proof:

Left to the pupil.

Discussion:

Show that $\triangle ABC : \triangle A'B'C' = a^2 : a'^2 = b^2 : b'^2$.

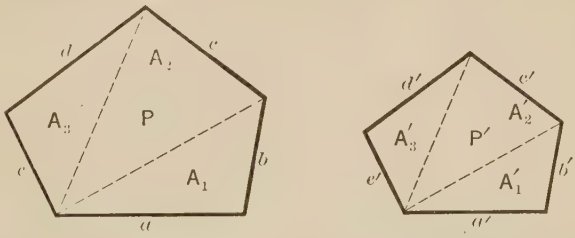
* **Ex. 1.** The areas of two similar triangles are to each other as the squares of any two corresponding altitudes or as the squares of any two corresponding medians. (See ex. 1, p. 232.)

Ex. 2. Two triangles are similar and the sides of one are 6″, 10″, 12″. If 4″ is the side in the second triangle corresponding to side 6″ in the first triangle, find the other sides.

Ex. 3. The area of a triangle is 72 sq. in. and one side is 24 in. If the corresponding side in a similar triangle is 16 in., find its area.

Proposition VII. Theorem

412. *The areas of two similar polygons are to each other as the squares of any two corresponding sides.*



Given: Similar polygons P and P', with a and a' any two corresponding sides.

To prove: $\frac{P}{P'} = \frac{a^2}{a'^2}$.

Analysis:

1. A possible line of proof is suggested by §§ 380 and 411. What set of proportions follows?
2. In what other form can the ratio $\frac{P}{P'}$ be written? (Ax. 6.)
3. Then from what series of equal ratios would this ratio come, by § 332?
4. For these series of ratios to be equal, how must the second ratios in each set in step 1 compare? Is this relation true? (See § 350 (2) and Ax. 14, p. 206.)

Proof:

Left to the pupil.

413. Corollary. *The areas of two similar polygons are to each other as the squares of their perimeters or as the squares of any two corresponding lines.*

Discussion:

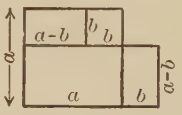
Show that $P : P' = b^2 : b'^2 = c^2 : c'^2 = \cdots$.

Ex. 1. If a, b, and c are any three lines, prove geometrically: $a(b + c) = ab + ac$.

Ex. 2. If a and b are any two lines, prove geometrically: $(a + b)^2 = a^2 + 2\,ab + b^2$.

Ex. 3. If a and b are any two lines $(a > b)$, prove geometrically: $(a - b)^2 = a^2 - 2\,ab + b^2$.

Ex. 4. If a and b are any two lines $(a > b)$, prove geometrically: $(a + b)(a - b) = a^2 - b^2$.



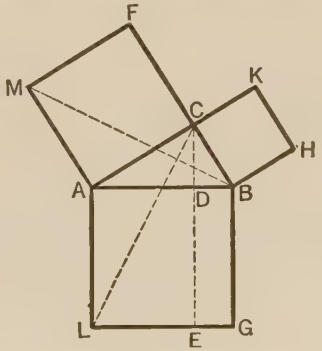
414. In § 369 was shown the relation that must exist between three numbers if they represent the sides of a right triangle. This relation states that the square of the number which represents the hypotenuse equals the sum of the squares of the numbers which represent the two legs. The proof there was algebraic. For example, in place of $(AC)^2$ the product $AB \times AD$ was substituted, the *numerical* equality of these two expressions being considered instead of the equality of the two geometric figures represented by them.

Proposition VIII will show the corresponding relation that must exist between the squares constructed on the three sides of a right triangle. The proof there will be the same as in § 369 excepting that it will be made *geometric* by representing the products by corresponding areas. For example $(AC)^2$ will be represented by the square constructed on line AC; $AB \times AD$ will be represented by the rectangle whose dimensions are AB and AD, and these two figures will be proved equal by showing that they represent equal areas.

Inasmuch as the numerical results are the same for both statements, the difference being in the method of proof resulting from the double interpretation that can be placed on $(AB)^2$ where AB represents a line, the two statements will be considered equal. Hence either proof may be used.

Proposition VIII. Theorem

415. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs.*



Given: Rt. $\triangle ABC$, with BL the square on the hypotenuse AB, and AF and BK the squares on the two legs.

To prove: $BL = AF + BK$, that is, $(AB)^2 = (AC)^2 + (BC)^2$.

Analysis:

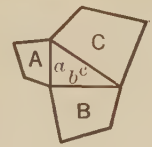
1. See § 369. What corresponding areas are represented by the statement $(AC)^2 = AB \times AD$, that is, $AL \times AD$? How are they constructed in the figure?
2. Now show (*a*) sq. AF = rect. AE, then (*b*) sq. BK = rect. BE.
3. Compare sq. AF and $\triangle AMB$; rect. AE and $\triangle ALC$. (See § 407.)
4. Then to make sq. AF = rect. AE, what relation must be shown between these two $\triangle$? Is this relation true? (See § 110.)
5. What cons. would show sq. BK = rect. BE?

Proof:

Left to the pupil.

416. Corollary 1. *The square on one leg of a right triangle is equal to the square on the hypotenuse diminished by the square on the other leg.*

417. Corollary 2. *If similar polygons are constructed on the three sides of a right triangle, the sum of the polygons on the legs is equal to the polygon on the hypotenuse.*



To prove: $A + B = C$.

Analysis:

1. $A + B = C$ if $\frac{A}{C} + \frac{B}{C} = 1$. Why?
2. What does $\frac{A}{C}$ equal? $\frac{B}{C}$ equal? Why?
3. Is their sum equal to 1? Why?

Discussion:

Is FCB a straight line? Why? Is a consideration of this question necessary in the proof? Explain. Likewise for ACK.

Historical Note. The proof of Proposition VIII is supposed to have been discovered originally by Pythagoras (about 550 B.C.), a Greek mathematician and teacher. For this reason it is called the *Pythagorean Theorem.*

The proof as given here and as found in every modern textbook is the one found in Euclid's Elements and was Euclid's own invention. It is therefore more than 2200 years old. It is not definitely known what demonstration Pythagoras gave to this famous theorem, but it is thought to have been either the one used in § 369 or the one given for Fig. 1, page 266.

Over twenty-five different proofs for this theorem have been discovered. Most of these consist of ingenious sectionings and rematchings of the areas of the three squares. The one given in this text, aside from its historic interest, is perhaps the simplest of those proofs which are purely geometric.

In Fig. 1, if from the square AB the four equal triangles a, b, c, d are taken, the square KL remains; if from the square AB the two rectangles CD and DE are taken, the squares AD and DB remain. Since the four triangles a, b, c, d equal the two rectangles, the two remainders are equal.

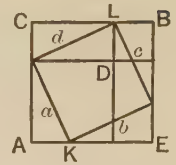


Fig. 1

The interesting proof which readily follows from Fig. 2 was worked out by President James A. Garfield and printed about 1876.

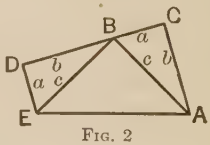


Fig. 2

Ex. 1. Prove Th. VIII, using Fig. 2.

Ex. 2. The legs of a right triangle are 36 in. and 8.1 in. Find the area of the square on the hypotenuse.

Ex. 3. The hypotenuse of a right triangle is 52″ and one leg is 20″. Find the area of the square constructed on the other leg.

Ex. 4. On sides AC and BC of any triangle, ABC, any parallelograms, $ACDE$ and $CBKF$, are drawn. ED and KF are produced to meet at P, the line PC is drawn, and parallelogram $ABML$ is drawn on AB with AL equal to and parallel to PC. Then

$\square ACDE + \square CBKF = \square ABML$.

Sug. Show $\square ACDE = \square LRNA$.

This extension of the Pythagorean theorem is due to the Greek geometer, Pappus (300 A.D.).

Ex. 5. The area of triangle ABC = 36 sq. in. If AC = 8 in. and AD = 3 in., D being in AC, find the area of triangle ABD.

Ex. 6. ABC is an isosceles triangle, base AB = 16 in., and altitude on AB = 6 in. If BC is extended through C to D so that CD = 4 in., find the area of triangle CAD.

Ex. 7. Corresponding sides in two similar pentagons are 4 in. and 6 in. The area of the first is 32 sq. in. Find the area of the second.

Ex. 8. Prove that two corresponding sides of two similar polygons are to each other as the square roots of their areas.

Ex. 9. The areas of two similar hexagons are 64 and 324. If a side of the second hexagon is 9, find the corresponding side in the first.

Exercises — Group 83

IMPORTANT THEOREMS INVOLVING LINES AND AREAS

* **1.** If a is one side of a square, the diagonal is $a\sqrt{2}$.

* **2.** If a is one side of an equilateral triangle, the altitude is $\frac{a}{2}\sqrt{3}$.

* **3.** If h is the altitude of an equilateral triangle, one side is $\frac{2}{3}h\sqrt{3}$.

* **4.** If a is one side of an equilateral triangle, the area is $\frac{a^2}{4}\sqrt{3}$.

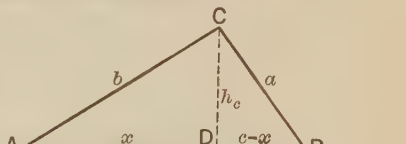
* **5.** If a is one side of a square, the area (K) is a^2.

6. Prove that the square on the diagonal of a square is equal to twice the area of the square.

* **7.** The area of a rhombus equals one half the product of the two diagonals.

* **8.** The area of any circumscribed polygon is equal to one half the product of its perimeter by the radius of the circle.

* **9.** If a, b, and c represent the sides of a triangle and $2s = a+b+c$, then the area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$.



Area of $\triangle ABC = \frac{1}{2}c \times h_c$.

$$h_c^2 = b^2 - x^2 = b^2 - \left[\frac{(b^2+c^2-a^2)}{2c}\right]^2.$$ (See ex. 3, p. 247.)

$$= \frac{1}{4c^2}[4b^2c^2 - (b^2+c^2-a^2)^2]$$

$$= \frac{1}{4c^2}(b+c+a)(b+c-a)(a-b+c)(a+b-c)$$

$$\therefore h_c = \frac{1}{2c}\sqrt{(b+c+a)(b+c-a)(a-b+c)(a+b-c)}$$

$\therefore$ Area of $\triangle ABC =$

$$\tfrac{1}{4}\sqrt{(b+c+a)(b+c-a)(a-b+c)(a+b-c)}$$

Since $a+b+c = 2s$, then $b+c-a = 2s-2a = 2(s-a)$, etc.

$\therefore$ Area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$.

* **10.** Show that in any triangle $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$.

Exercises — Group 84

OTHER THEOREMS INVOLVING AREAS

1. $ABCD$ is a parallelogram and R, S any points in CD and AD respectively.

Prove: $\triangle SBC = \triangle RAB$.

2. $ABCD$ is a parallelogram and E and F are the mid-points of BC and AD respectively. If P is any point in EF, prove that $\triangle PAB = \frac{1}{4}\square ABCD$.

3. The non-parallel sides of a trapezoid form with the diagonals two equal triangles.

* **4.** The line joining the mid-points of the bases of a trapezoid divides the trapezoid into two equal parts.

5. The line joining the mid-points of two adjacent sides of a parallelogram cuts off a triangle which is one eighth of the parallelogram. (See § 411.)

6. If each diagonal of a quadrilateral divides the quadrilateral into two equal parts, the diagonals bisect each other.

7. State and prove the converse of ex. 6.

8. If P is any point in the median CD of $\triangle ABC$, prove: $\triangle PAC = \triangle PBC$.

9. If D is the mid-point of AB and E the mid-point of AC of triangle ABC, prove: $\triangle ABE = \triangle ADC$.

10. D and E are the mid-points of sides AC and BC of triangle ABC. Through D and E any two parallels are drawn meeting AB (produced if necessary) in K and L respectively.

Prove: $\square KLED = \frac{1}{2}\triangle ABC$.

11. If any point within a parallelogram is joined to the four vertices, the sum of either pair of opposite triangles is equal to one half the parallelogram.

12. If through any point in a diagonal of a parallelogram parallels to the sides are drawn, four parallelograms are formed of which the two on opposite sides of the diagonal are equal.

13. The lines joining the mid-points of the sides of a quadrilateral, taken in order, form a parallelogram equal to one half the quadrilateral.

14. If a square and a rhombus are constructed on the same base, which has the greater area? Prove.

15. Lines drawn from any point in one side of a triangle to the mid-points of the other two sides form a quadrilateral which is equal to one half the given triangle.

* **16.** If M, the mid-point of either leg of a trapezoid $ABCD$, is joined to the opposite vertices, the middle triangle formed is equal to one half the trapezoid.

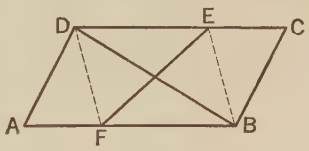
For example, prove: $\triangle MAD = \frac{1}{2}$ trap. $ABCD$; that is, $\triangle MAB + \triangle MCD = \frac{1}{2}$ trap. $ABCD$.

* **17.** The area of a trapezoid is equal to the product of one leg by the perpendicular on that leg from the mid-point of the other leg.

* **18.** A straight line through the mid-point of a diagonal of a parallelogram, terminating in two opposite sides, divides the parallelogram into two equal parts.

In this case are the parts also congruent?

19. Prove that a straight line which bisects a parallelogram passes through the intersection of its diagonals. (Converse of ex. 18.)



Given: EF bisects parallelogram $ABCD$.

Prove: EF passes through the mid-point of DB.

Sug. Show that $FBED$ is a parallelogram.

Exercises — Group 85

Some Numerical Properties of Lines and Areas

1. If the diagonals of a quadrilateral are perpendicular to each other, the sum of the squares on one pair of opposite sides equals the sum of the squares on the other pair of opposite sides.

2. M is the mid-point of leg AC of the right triangle ABC and AB is the hypotenuse.

Prove: $(AB)^2 - (BM)^2 = 3\,(CM)^2$.

3. ABC is an isosceles triangle ($AC = BC$), and a perpendicular to BC at B meets CA produced at O.

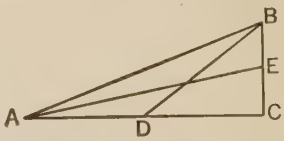
Prove: $(OB)^2 = (OA)^2 + 2\,OA \times AC$.

*4. The line joining the mid-points of two sides of a triangle cuts off a triangle whose ratio to the original triangle is 1 : 4.

5. If an equilateral triangle is constructed on the hypotenuse of a right isosceles triangle, find the ratio of the two areas.

6. Find the ratio between the areas of two equilateral triangles if the base of the first is the altitude of the second.

7. In the figure, D and E are the mid-points of legs AC and BC of the right triangle ABC.



Prove: $4\,(BD)^2 + 4\,(AE)^2 = 5\,(AB)^2$.

8. If D is any point in the base AB of isosceles triangle ABC, prove: $(CD)^2 = (AC)^2 - AD \times DB$.

9. Two chords of a circle, AB and CD, are perpendicular to each other and intersect in P.

Prove: $(PA)^2 + (PB)^2 + (PC)^2 + (PD)^2 =$ (a diameter)^2.

Sug. Draw diameter BE and show that chord $AC =$ chord ED, etc.

Exercises — Group 86

Numerical

1. Find the area of an isosceles triangle if the base is 10″ and one of the legs is 13″.

2. One side of an equilateral triangle is 6″. Find the area.

3. An altitude of an equilateral triangle is 9″. Find one side and the area.

4. Find the area of a rectangle if the diagonal is 24″ and one side is 18″.

5. A diagonal of a rectangle is 16″ and it makes an angle of 30° with the base. Find the sides of the rectangle.

6. Two similar polygons have corresponding sides 15 in. and 6 in. If the area of the first is 2700 sq. in., find the area of the second.

7. Two similar polygons have areas 400 sq. in. and 625 sq. in. If a side of the first is 15 in., find the corresponding side of the second.

8. Two similar triangles have areas 400 sq. in. and 256 sq. in. If an altitude of the first is 32 in., find the corresponding altitude of the second. What are the two corresponding bases used?

9. A circle with a radius of 6″ is inscribed in a quadrilateral whose sides are 16″, 12″, 8″, 10″. Find the area of the quadrilateral.

10. A diagonal of a square is 20 in. Find the area of the square.

11. The perimeter of a rhombus is 104 in. and one diagonal is 48 in. Find the area of the rhombus.

12. The area of an equilateral triangle is 48 sq. in. Find one side.

13. One side of an equilateral triangle is 12 in. Find the side of the equilateral triangle that has twice the area.

14. Find the length of the chord of a quadrant of a circle whose radius is 12 in.

15. The area of a polygon is 64 sq. in. Find the area of a similar polygon if two corresponding sides of the two polygons are as 1 to 2.

16. One side of a polygon is 6 in. Find the corresponding side in a similar polygon which is four times as large; twice as large; one third as large.

17. One leg of a right triangle is 5 in. and the altitude on the hypotenuse is 3 in. Find the area of the triangle.

18. The area of a rhombus is 600 sq. in. and the diagonals are in the ratio 3 : 4. Find the diagonals and one side.

19. In triangle ABC, $AB = 8$ in. and $AC = 10$ in. Find the area of the triangle (a) if $\angle A = 60°$; (b) if $\angle A = 45°$.

20. CA and DB are each perpendicular to AB, and $CA = 5.5$ in., $DB = 10.5$ in., $AB = 12$ in. Find the length of line CD when CA and DB are on (a) the same side of AB; (b) opposite sides of AB.

21. One side of a rectangle is 16 in. and the diagonal is 34 in. Find a side of the equilateral triangle whose area equals the area of the rectangle.

22. If the radius of a circle is 12 in., find the length of a chord of an arc of 45°; 30°; 60°.

23. In ex. 2, p. 235, find the ratio of the two areas into which DE divides the triangle ABC.

24. The altitude of a triangle is 12 in. How far from the vertex must a parallel to the base be drawn to cut off a triangle whose area is one third of the given triangle?

25. Find the area of an isosceles trapezoid in which the shorter base is 8 in., one leg is 12 in., and one angle 30°.

26. In the trapezoid $ABCD$ base $AB = 18$ in., $AD = 8$ in., $\angle A = 60°$, $\angle B = 45°$. Find the area of the trapezoid.

27. One angle of a parallelogram is 60°. If the shorter side is 8 in. and the shorter diagonal 13 in., find the area of the parallelogram.

28. Find the area of a quadrilateral $ABCD$ if $AB = 16$ in., $BC = 20$ in., $CD = 13$ in., $DA = 17$ in., $AC = 24$ in.

29. The sides of a triangle are 24 in., 36 in., and 48 in. Find the area of the triangle and the altitude on side 36 in.

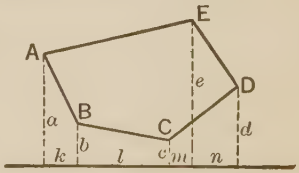
30. The sides of a triangle are in the ratio $4:5:7$. If the area of the triangle is $100\sqrt{6}$ sq. in., find the three sides.

31. The bases of a trapezoid are 30 in. and 9 in. and the non-parallel sides are 10 in. and 17 in. Find the area of the trapezoid.

32. The diagonals of a parallelogram are 16 in. and 30 in. and the angle between them is 120°. Find the area and the sides.

33. The diagonals of a parallelogram are 30 in. and 28 in. and the area is 336 sq. in. Find the sides of the parallelogram. (See ex. 6, p. 258.)

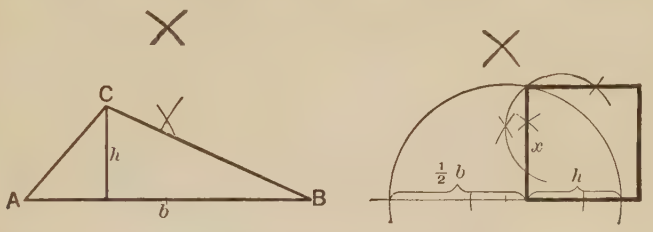
34. Find the area of field $ABCDE$ from the following measurements, expressed in rods:
$a = 18, b = 7, c = 4, d = 13, e = 24,$
$k = 5, l = 16, m = 3, n = 7$



35. The following rule is given for finding the diagonal of a square: "Multiply one side by 10, take away 1% of this product, and divide the remainder by 7." Test the accuracy of this rule in a square whose side is 24.

Proposition IX. — Problem

418. *To construct a square equal to a given triangle.*



Given: $\triangle ABC$ with base b and altitude h.

Required: To construct a square whose side is x, such that $x^2 = \frac{1}{2}\, b \cdot h$.

Analysis:

1. Write a proportion from which this equation would come.
2. Hence what cons. would determine x? (See § 385.)

Construction and Proof:

Left to the pupil.

* **Ex. 1.** To construct a square equal to a given parallelogram; a given trapezoid.

Ex. 2. To construct a square equal to a given part of a given square.

Sug. If a is one side of the given square, $m : n$ the given ratio, and x one side of the required square, then $x^2 = \frac{m}{n}\, a^2$; that is, $x = \frac{a}{n}\sqrt{mn}$. Cons. $k = \sqrt{mn}$, and then cons. $x = a \cdot \frac{k}{n}$.

Ex. 3. To construct a polygon similar to a given polygon and having a given ratio to it.

Sug. Here $\frac{n}{m} = \frac{a^2}{x^2}$. Determine x as in ex. 2, and use § 386.

*Ex. 4. To construct a parallelogram equal to a given square (a^2) and with the sum of its base and altitude equal to a given line s.

Sug. 1. Let x = alt. of the $\square$. Then $s - x$ = base of the $\square$.

2. $\therefore x(s - x) = a^2$, or $sx - x^2 = a^2$. That is, $a^2 + x^2 - sx = 0$.

3. $\therefore a^2 + \left(x - \frac{s}{2}\right)^2 = \left(\frac{s}{2}\right)^2$, or $\left(x - \frac{s}{2}\right)^2 = \left(\frac{s}{2}\right)^2 - a^2$.

4. Find $\left(x - \frac{s}{2}\right)$ by § 416; then find x and $s - x$.

Any $\square$ with these dimensions would be the required $\square$.

Since $a^2 = x(s - x)$, a is the mean proportional between x and $s - x$. Hence x and $s - x$ can also be found by § 368.

Ex. 5. To construct a parallelogram equal to a given square (a^2) and with the difference between its base and altitude equal to a given line d.

419. *To transform a plane figure* means to construct another figure of different form but of equal area. The transformation of plane figures is usually accomplished by means of Th. III, Cor. 3. Frequently two or more transformations are necessary before the desired form is obtained.

Fig. 1

Fig. 2

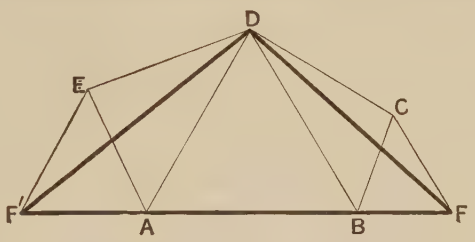
Fig. 3

If to the triangle ABC the triangle BCD is added, as in Fig. 1, a *quadrilateral* $ABDC$ results. If, however, to the triangle ABC a triangle BFC is added, as in Fig. 2, a *triangle* AFC results. If then the triangles DBC and FBC are equal, the quadrilateral $ABDC$ equals the triangle AFC.

In Fig. 3, therefore, triangle AFC equals quadrilateral $ABDC$ if $\triangle DBC = \triangle FBC$, or, since they have the same base, if their altitudes on BC are equal. This relation in turn is true if DF is parallel to BC. Since BC is a diagonal of the quadrilateral and D a vertex, this suggests a means of locating point F and thereby transforming the quadrilateral into a triangle.

Proposition X. Problem

420. *To transform a given polygon into a triangle.*



Given: Any polygon *ABCDE*.

Required: To transform *ABCDE* into a triangle.

Analysis:

1. Reduce the polygon one side by transforming quad. *ABCD* into $\triangle AFD$, giving polygon *AFDE*.
2. Reduce *AFDE* one side in a similar way, giving $\triangle F'FD \cdots$

Construction and Proof:

Left to the pupil.

421. Corollary. *To transform a given polygon into a square.*

Use § 420, then § 418.

Discussion:

1. In the first transformation, what are the two equal triangles and to what common area is each added? in the second transformation?
2. Show how F' might have been located on the same side of the polygon as *F*.
3. Draw a polygon of six sides and by successive transformations change it into a triangle.

Make a good-sized polygon and at each transformation note the two equal triangles and the common area to which each is added.

* **Ex.** To construct a polygon T similar to a given polygon R and equal to another given polygon S.

Sug. Transform R and S into squares with sides r and s. Then if x is a side of T corr. to side a of R, $\frac{a^2}{x^2} = \frac{R}{T} = \frac{r^2}{s^2}$; that is, $\frac{r}{s} = \frac{a}{x}$. Hence find x and use § 387.

Exercises — Group 87

Miscellaneous Transformations and Constructions

Construct a triangle:

1. Having a given line s for its base and equal to a given triangle.

Sug. If x = the altitude of the required $\triangle$, then $b \cdot h = s \cdot x$.

2. Having a given line for its base and equal to a given square.

3. Similar to a given triangle and 4 times as large; 3 times as large.

Transform a given triangle ABC into:

4. A right triangle which has a given leg; which is isosceles.

5. An isosceles triangle with base AB; with base a given line m.

6. Another triangle having two of its sides given.

Sug. Use one of the given lines as the base.

Construct a square:

* **7.** Equal to the sum of two given squares. (See Th. VIII.)

* **8.** Equal to the difference of two given squares.

9. Equal to the sum of three given squares.

10. Equal to 3 times a given triangle.

Sug. Here $x^2 = \frac{3}{2}\, a \cdot b$. $\therefore 2x = \sqrt{6\, a \cdot b}$.

11. Equal to 3 times a given square; equal to three fourths of a given square.

12. In ex. 1, would it be possible to require that the triangle to be constructed be right? isosceles? that it have a base angle of 60°? that it have an adjacent side of a given length? Illustrate.

13. Transform a given parallelogram into:

(*a*) A rectangle having one side equal to a given line.

(*b*) Another parallelogram having a given side and a given angle.

14. Divide a given triangle ABC into three equal parts by two lines from C.

15. Divide a given triangle ABC into two equal parts by a line parallel to AB.

* **16.** Find a point O in triangle ABC such that the triangles OAB, OBC, and OCA are to each other as $3:4:5$.

SUG. See § 405 (*a*).

17. Divide a parallelogram into two equal parts by a line: (*a*) parallel to one side; (*b*) perpendicular to one side.

18. Transform a given triangle ABC into:

(*a*) An isosceles triangle with one leg equal to a given line m.

SUG. Use m as the base and find the altitude.

(*b*) An equilateral triangle. (See ex. top of page 276.)

* **19.** To construct a polygon T equal to the sum of two similar polygons R and S and similar to them. (See §§ 417 and 387.)

20. Construct a triangle similar to a given triangle and equal to a given parallelogram. (See ex. top of p. 276.)

21. To bisect a trapezoid by a line drawn from any point in the shorter base. (See ex. 4, p. 268.)

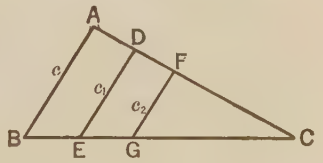
22. To bisect a trapezoid by a line through a vertex.

23. To divide a triangle ABC into two equal parts by a line from any point P in AB. (See § 410.)

24. To divide a parallelogram $ABCD$ into three equal parts by lines from vertex A.

SUG. Let the lines be AP and AP'. Then $\triangle ADP = \frac{1}{3}\,\square ABCD = \frac{2}{3}\,\triangle ADC$, etc.

25. To divide a triangle into three equal parts by lines parallel to AB.



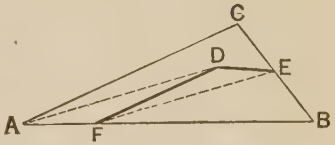
SUG. $\dfrac{\triangle CDE}{\triangle CAB} = \dfrac{{c_1}^2}{c^2} = \dfrac{2}{3}$. Why?

$\therefore \sqrt{3} : \sqrt{2} = c : c_1$. Hence find c_1, etc.

Exercises — Group 88

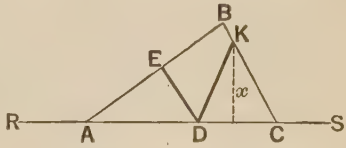
APPLICATIONS

1. A well is situated at D in a triangular pasture ABC. A farmer wishes to run fences from the well so as to form two pastures of the same area. Show how he can do it.



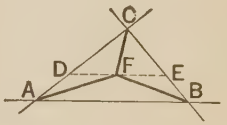
SUG. From E, the center of BC, draw $EF \parallel AD$. Prove that EDF is the fence required.

2. ABC is a triangular pasture fronting on a straight road RS. A farmer wishes to run fences from a point D on RS to the other two sides, forming three pastures of equal area with a common gate at D, which is opposite his barn. How can he do it?



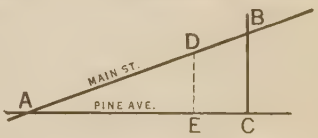
SUG. Locate K and E by the use of § 404.

3. ABC is a city lot formed by the intersections of three streets. It is desired to divide this lot into three equal lots each having a frontage on one of the streets. Show how it can be done.



SUG. On one side, as AC, take $AD = \frac{1}{3}\,AC$ and draw $DE \parallel AB$. Lines from A, B, and C to F, the mid-point of DE, form the desired division. Why?

4. Two streets intersect at an angle A. It is desired to "part off" a triangular tract ABC of given size to be used as a park by cutting a street CB through at right angles to one of the streets. Find AC.



SUG. Form any rt. $\triangle ADE$ and determine its area. Then $AC = AE\sqrt{\triangle ABC \div \triangle ADE}$. Why?

5. Required to part off from a corner of a field a triangular space of given area by a line starting from a given point P in one of the sides.

BOOK FIVE

REGULAR POLYGONS. CIRCLES

REGULAR POLYGONS

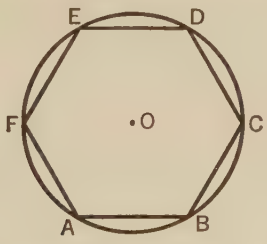
422. A *regular polygon* is a polygon which is both equiangular and equilateral. (§ 58.)

To prove a given polygon regular, we must show:

1. *That all the angles of the polygon are equal.*
2. *That all the sides of the polygon are equal.*

Proposition I. Theorem

423. *An equilateral polygon inscribed in a circle is regular.*



Given: Polygon $ABCD \cdots$ inscribed in circle O, with $AB = BC = CD \cdots$.

To prove: Polygon $ABCD \cdots$ is regular.

Analysis:

1. What remains to be shown to make the polygon regular?
2. How do the arcs formed in the figure compare? Why?
3. In a polygon of n sides, on how many arcs does each $\angle$ of the polygon stand? Why?
4. Hence why are the $\measuredangle$ of the polygon equal?

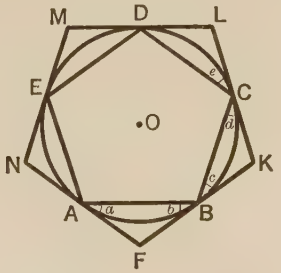
Proof: Left to the pupil.

Proposition II. Theorem

424. *If a circle is divided into any number of equal parts:*

(*a*) *The chords joining the successive points of division form a regular polygon inscribed in the circle.*

(*b*) *The tangents drawn at the points of division form a regular polygon circumscribed about the circle.*



Given: Circle O divided into n equal parts and A, B, C $\cdots$, the points of division joined in order by chords AB, BC $\cdots$; also the tangents at A, B, C $\cdots$ intersecting in F, K, L $\cdots$.

To prove: (*a*) ABC $\cdots$ is a regular inscribed polygon.

(*b*) FKL $\cdots$ is a regular circumscribed polygon.

(*a*) **Analysis**:

1. By § 423 what condition would make ABC $\cdots$ a regular polygon?
2. Is this condition true in the drawing? Why?

Proof:

For both (*a*) and (*b*) left to the pupil.

(*b*) **Analysis**:

1. In terms of the drawing what must be proved to make FKL $\cdots$ a regular polygon?
2. Show this by proving $\triangle AFB \cong \triangle BKC \cong \triangle CLD$ $\cdots$ (§ 113), from which $\angle F = \angle K = \angle L$ $\cdots$ and $FB = BK = KC = CL$ $\cdots$,

425. COROLLARY 1. *If the mid-points of the arcs subtended by the sides of a regular inscribed polygon are joined to the adjacent vertices, a regular inscribed polygon of double the number of sides is formed.*

426. COROLLARY 2. *If tangents are drawn at the mid-points of the arcs of adjacent points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon of double the number of sides is formed.*

Ex. 1. Show that the center of the inscribed or the circumscribed circle of a regular polygon is the same distance from all the sides.

Ex. 2. If in a polygon a point can be found equidistant from all the sides, prove that a circle may be inscribed in that polygon.

Ex. 3. Under the conditions found in ex. 2, could a circle be circumscribed about the polygon? Explain.

Ex. 4. Which of the polygons requires only *one* of the conditions in § 422 to make it regular? In this case, does the other condition follow as a necessary consequence? Explain.

Ex. 5. The square and the equilateral triangle are each regular polygons.

Ex. 6. Is an equilateral polygon regular if the polygon is a triangle? a quadrilateral? a hexagon? Illustrate.

Ex. 7. Is an equiangular polygon regular if the polygon is a triangle? a quadrilateral? a hexagon? Illustrate.

427. Since a division of a circle into equal parts will result from a division into equal parts of the angles about the center, § 424 makes it possible to construct geometrically an inscribed or a circumscribed regular polygon of n sides when $\frac{360^\circ}{n}$ is an angle that can be constructed with ruler and compasses. In cases where $\frac{360^\circ}{n}$ is an angle which cannot be constructed geometrically, the protractor may be used. (See § 58.)

Proposition III. Theorem

428. *A circle may be circumscribed about any regular polygon, and a circle may also be inscribed in it.*

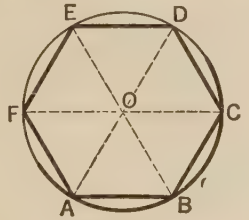


Fig. 1

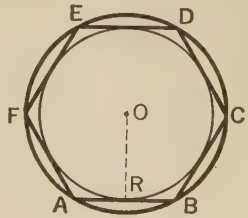


Fig. 2

Given: Regular polygon $ABCD \cdots$.

To prove: (*a*) A circle may be circumscribed about $ABCD \cdots$ (Fig. 1).

(*b*) A circle may be inscribed in $ABCD \cdots$ (Fig. 2).

(*a*) **Analysis:**

1. A circle may at once be passed through three consecutive vertices, A, B, C. Explain. Represent the center and the radii to these points.
2. What condition in the drawing would make the circle pass through the next consecutive vertex?
3. Show that this condition is true by proving $\triangle OAB \cong \triangle OCD$. (See § 110.)

Proof:

For both (*a*) and (*b*) left to the pupil.

(*b*) **Analysis:**

1. This circle is possible if there is a point within the polygon equidistant from the sides. Explain.
2. Draw the circumscribed circle and show that there is such a point.
3. What is the center and the radius of the required circle?
4. Why are the sides tangent to the circle? (See § 246.)

Discussion:

1. Where in (*a*) and (*b*) is the fact used that $ABCD \cdots$ is regular? Are both conditions for a regular polygon used?
2. Through what points in the sides of the regular polygon does the inscribed circle pass? Why?

Ex. 1. A circle may be circumscribed about or inscribed in any square or equilateral triangle.

* **Ex. 2.** The circumscribed and inscribed circles of a regular polygon are concentric.

429. The *center of a regular polygon* is the common center of the inscribed and circumscribed circles. (See point O, Fig. 2, § 428.)

430. The *radius of a regular polygon* is the radius of the circumscribed circle. (See line OA, Fig. 1, § 428.)

431. The *angle at the center of a regular polygon* is the angle between the two radii drawn to the extremities of any side. (See $\angle AOB$, $\angle BOC$, etc., Fig. 1, § 428.)

432. The *apothem* of a regular polygon is the radius of the inscribed circle. (See line OR, Fig. 2, § 428.)

433. COROLLARY. *In any regular polygon of* n *sides:* (*a*) *the angle at the center equals* $\frac{360°}{n}$; (*b*) *any angle of the polygon equals* $\frac{(n-2) \times 180°}{n}$.

Ex. 1. How many sides has a regular polygon whose angle at the center is 30°?

Ex. 2. An exterior angle of a regular polygon is 45°. How many sides has the polygon?

Ex. 3. Is there a regular polygon whose angle at the center is $17\frac{1}{7}°$? 7° 30′?

Ex. 4. In a regular octagon find the number of degrees in (*a*) one of the angles; (*b*) one of the angles at the center.

Ex. 5. Same as ex. 4 for a regular pentagon; a regular hexagon.

Exercises — Group 89

1. An angle at the center of a regular polygon is bisected by the apothem drawn to the corresponding side.

* **2.** The radius drawn to any vertex of a regular polygon bisects the angle of the polygon.

* **3.** An angle of a regular polygon is equal to the supplement of the angle at the center.

* **4.** An exterior angle of a regular polygon is equal to the angle at the center.

5. The diagonals drawn from a vertex of a regular polygon of n sides divides the angle at that vertex into $(n - 2)$ equal parts.

6. The lines joining opposite vertices of a regular inscribed hexagon pass through the center of the circle.

7. In any regular polygon there is a point which is equidistant from the sides and which is also equidistant from the vertices.

8. The diagonals of a regular quadrilateral are perpendicular to each other.

9. The diagonals of any regular pentagon are equal.

SUG. Circumscribe a circle about the pentagon.

* **10.** If two diagonals of a regular pentagon intersect, the greater segment of each is equal to a side of the pentagon.

11. $ABCDE$ is a regular pentagon. AC and BE are drawn and the two lines intersect in P. Prove that $CDEP$ is a parallelogram.

12. Can any angle be used as the angle at the center of some regular polygon? Explain.

13. If squares are erected on the sides of a regular hexagon, the lines joining their exterior vertices form a regular polygon of 12 sides (duodecagon).

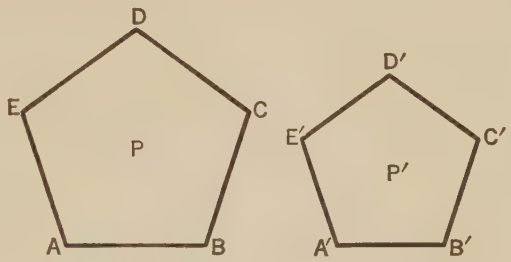
14. The lines joining the mid-points of the apothems of a regular pentagon form a regular pentagon.

* **15.** An equiangular polygon circumscribed about a circle is regular.

* **16.** A polygon is regular if its inscribed and circumscribed circles are concentric.

PROPOSITION IV. THEOREM

434. *Regular polygons of the same number of sides are similar.*



Given: Two regular polygons P and P', each having n sides.

To prove: $P \sim P'$.

Analysis:

1. By definition what must be proved to make two polygons similar?
2. What is the measure of each angle in P and P'? (§ 433 (*b*).)
3. Hence how do the angles of P and P' compare? Why?
4. Why is the other condition for similarity true?

Proof:

Left to the pupil.

Ex. 1. To what class of triangles does Th. IV apply? to what class of quadrilaterals? Why?

Ex. 2. If all diagonals of a regular pentagon are drawn, prove that 25 isosceles triangles are formed which are congruent in groups of 5.

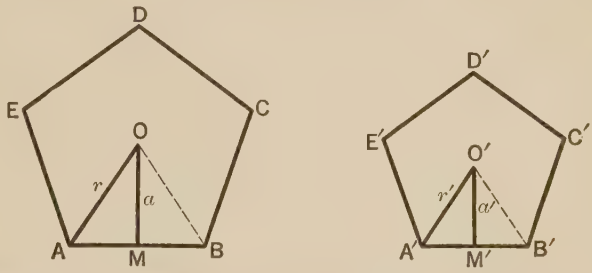
SUG. Circumscribe a circle about the pentagon.

Ex. 3. If all the diagonals of a regular pentagon are drawn, their intersections are the vertices of another regular pentagon.

Ex. 4. If the alternate sides of a regular inscribed polygon are produced to meet, the points of intersection are the vertices of another regular polygon.

Proposition V. Theorem

435. *The perimeters of two regular polygons of the same number of sides have the same ratio as their radii or as their apothems.*



Given: Two regular polygons $ABC\cdots$ and $A'B'C'\cdots$ having the same number of sides, with perimeters p and p', radii r and r', and apothems a and a' respectively.

To prove: $\frac{p}{p'} = \frac{r}{r'} = \frac{a}{a'}$.

Analysis:

1. What is the relation between these polygons, by § 434?
2. Hence what does the ratio $\frac{p}{p'}$ equal? (See § 379.)
3. How can this ratio be shown equal to $\frac{r}{r'}$? $\frac{a}{a'}$? (See § 363.)

Proof:

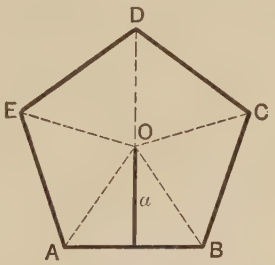
Left to the pupil.

436. Corollary 1. *The perimeters of two regular polygons of the same number of sides have the same ratio as any two corresponding sides.*

437. Corollary 2. *The areas of any two regular polygons of the same number of sides are to each other as the squares of their radii or as the squares of their apothems.*

Proposition VI. Theorem

438. *The area of a regular polygon is equal to one half the product of its perimeter and its apothem.*



Given: Regular polygon $ABC \cdots$ with perimeter denoted by p, apothem by a, and area by K.

To prove: $K = \frac{1}{2} a \times p$.

Analysis:

1. What area formula is suggested?
2. How should the polygon be divided to use this formula?
3. What would the sides of the polygon represent? the apothem?
4. Find the several areas and simplify their sum.

Proof:

Left to the pupil.

Discussion:

1. Why is a the altitude of *each* triangle?
2. By the method of the theorem above derive the formula for the area of a square; for the area of an equilateral triangle.

Exercises — Group 90

1. The sides of two regular octagons are 3 in. and 6 in. respectively. Find the ratio of their perimeters and their areas.

2. The corresponding sides of two regular pentagons are 2 in. and 7 in. What is the ratio of their perimeters? apothems? areas?

3. The perimeters of two regular polygons having the same number of sides are 48 in. and 72 in. A side of one is 6 in. What is the corresponding side of the other?

4. Find the ratio between the two perimeters, the two areas, and two corresponding apothems of two regular hexagons whose sides are $2\frac{3}{4}$ in. and $4\frac{1}{8}$ in.

5. One side of a regular hexagon is 6 in. Find its area.

6. One side of a regular polygon of n sides is 8 in. If its radius is r, find its area in terms of n and r.

7. The apothem of a regular hexagon is $5\sqrt{3}$ in. Find one of its sides and the area.

8. One side of a regular octagon is 12 in. Find its apothem and its area.

CIRCLES

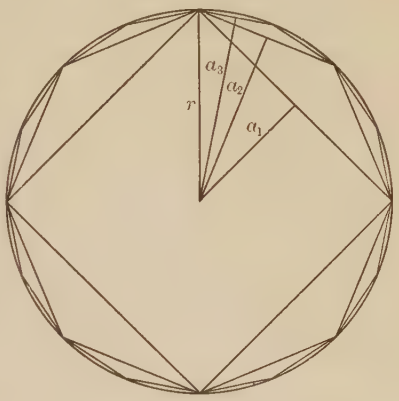
Measurement of the Circumference of a Circle

439. The method used in measuring any segment of a straight line is to select a unit straight line and apply it to the given line segment. (See §§ 41 and 264.) This method is apparently not applicable to finding the length of a circle, for it would be impossible to apply a unit straight line to a circle and determine how many of these units the circle contains.

440. A method for finding the approximate length of a circle, which is both accurate and geometric, will now be considered.

Inscribe a square in the given circle. Then form a regular inscribed polygon of double the number of sides. (See § 425.) Suppose that this process of doubling the number of sides of the inscribed regular polygon is continued indefinitely. It is apparent that the perimeter, the apothem, and the included area of the successive polygons are steadily increasing in magnitude and are getting closer in value to the circumference, the radius, and the area, respectively, of the circle.

Hence any relation between the perimeter, the apothem, and the area of the inscribed polygon is getting closer in value to the relation between the corresponding parts of the circle. The fact that these relations are true of the circle will be assumed here, and the following statement will be used as the basis for finding the circumference and the area of a circle:



441. *Any relation between the perimeters, the apothems, and the areas of regular polygons that remains true as the number of sides is successively doubled is true also between the corresponding parts of circles.*

If a circumscribed regular polygon were used in § 440 and the number of sides were repeatedly doubled (see § 426), the same inference could be made regarding the successive perimeters and areas.

From these considerations we infer without proof the following theorems:

THEOREM I. *If the number of sides of a regular inscribed polygon is indefinitely increased, the apothem approaches the radius of the circle as a limit.*

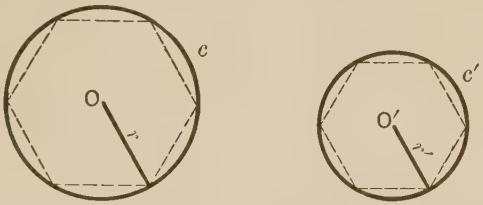
THEOREM II. *If the number of sides of a regular inscribed and of a regular circumscribed polygon is indefinitely increased:*

(*a*) *their perimeters approach the circumference of the circle as a limit;*

(*b*) *their areas approach the area of the circle as a limit.*

Proposition VII. Theorem

442. *The circumferences of two circles are in the same ratio as their radii.*



Given: Two circles O and O', with circumferences denoted by c and c' and radii by r and r' respectively.

To prove: $\frac{c}{c'} = \frac{r}{r'}$.

Analysis:

1. What corr. proportion for regular polygons was established in § 435?
2. What cons. does this suggest in the diagram? Is this cons. possible? Why?
3. Then is the desired proportion true? (See § 441.)

Proof: Left to the pupil.

443. Corollary 1. *The circumferences of two circles are in the same ratio as their diameters.*

That is, $\frac{c}{c'} = \frac{d}{d'}$.

444. Corollary 2. *The ratio of any circumference to its diameter is constant.*

1. This means that the ratio is the same for any two circles.
2. To what is the ratio $\frac{c}{d}$ equal, from Cor. 1?
3. Then is the condition in step 1 satisfied?

445. This constant is represented by the symbol π(pi), one of the letters in the Greek alphabet.

446. COROLLARY 3. *The circumference of any circle is expressed by the formula* $c = \pi d$, or $c = 2\pi r$.

447. COROLLARY 4. *The length of an arc of a circle, in linear units, is to the circumference as the number of degrees in the arc is to* 360.

448. In § 478 it will be shown that $\pi = 3\frac{1}{7}$ approximately, or, more accurately, $\pi = 3.1416$.

Exercises — Group 91

Use $\pi = \frac{22}{7}$ in the following exercises.

Find the circumference of a circle if:

1. $r = 7$ in. **3.** $r = 3.5$ yd. **5.** $r = m$ in.

2. $r = 28$ in. **4.** $d = 6\sqrt{2}$ in. **6.** $d = 12.48$ ft.

Find the diameter of a circle if the circumference is:

7. 88 in. **9.** 300 in. **11.** 15.27π in.

8. 26.4 ft. **10.** m yd. **12.** 17.28 ft.

Find the length of an arc when:

13. Its angle is 50°, $r = 12$ in. **15.** Its angle is 72°, $d = 8$ yd.

14. Its angle is 45°, $r = 8$ ft. **16.** Its angle is 135°, $r = 5.4$ in.

17. If the length of a quadrant is 8 ft., what is the radius?

18. Find the angle at the center of a circle having an arc 10 ft. long if the radius is 15 ft.

MEASUREMENT OF THE AREA OF A CIRCLE

449. By the *area of a circle* is meant the area of the surface inclosed by the circle.

450. The method used in § 391 is not applicable to finding the area of a circle. Accordingly a process similar to that described in § 440 is generally used to get an approximate measure of the area of a circle, and the principle given in § 441 is used to determine the corresponding formula.

Proposition VIII. Theorem

451. *The area of a circle is equal to one half the product of its circumference and its radius.*

Given: Circle O, with radius r, circumference c, and area K.
To prove: $K = \frac{1}{2} c \times r$.

Analysis:

1. What corr. relation for regular polygons was established in § 438?
2. What cons. does this suggest in the diagram? Is this cons. possible? Why?
3. Then why is the desired relation true?

Proof:

Left to the pupil.

452. Corollary 1. *The area of any circle is expressed by the formula* $K = \pi r^2$, or $K = \frac{1}{4}\pi d^2$.

To what is c equal? Substitute this value.

453. Corollary 2. *The areas of two circles are to each other as the squares of their radii or as the squares of their diameters.*

Prove: $\frac{K}{K'} = \frac{r^2}{r'^2} = \frac{d^2}{d'^2}$.

454. A *sector of a circle* is the figure formed by two radii and the arc intercepted by them. (See figure OAB in § 227.)

455. In the same circle or in different circles *similar sectors*, *similar arcs*, and *similar segments* are sectors, arcs, and segments that correspond to equal central angles.

456. COROLLARY 3. *The area of a sector of a circle is to the area of the circle as the angle of the sector is to* 360°.

457. COROLLARY 4. *The area of a sector is equal to one half the product of the radius and the length of the arc.*

What is the formula for the length of an arc?

Use this in Cor. 3.

458. COROLLARY 5. *The area of a segment of a circle whose arc is a minor arc is equal to the area of the corresponding sector minus the area of the triangle formed by the two radii and the chord of the segment.*

459. COROLLARY 6. *Similar sectors are to each other as the squares of their radii.*

Exercises — Group 92

Use $\pi = \frac{22}{7}$ in the following exercises.

Find the area of a circle in which:

1. $r = 7$ ft. **3.** $r = 35$ in. **5.** $d = .042$ yd.

2. $d = 4.2$ ft. **4.** $r = 1\frac{3}{4}$ ft. **6.** $r = .63$ ft.

Find the radius of the circle if the area is:

7. 324 sq. in. **8.** 55.44 sq. ft. **9.** $25\ m^2\pi$ sq. yd.

Find the area of a circle if its circumference is:

10. 52.8 ft. **11.** $24\ \pi$ in. **12.** C yd.

Find the area of a circular sector if:

13. Its arc is 30° and $r = 18$ in.

14. Its arc is 45° and $r = 3.2$ yd.

15. Its arc is 150° and $r = 10$ in.

16. In a circle whose radius is 12 in. find the area of a sector whose arc is 18 in. long. How many degrees are there in the angle of the sector?

17. Find the area of the segment of a circle if the arc of the segment is 150° and r equals 18 in.

18. Two circles of radii 3 in. and 4 in. have in them similar sectors. If the area of the first sector is 24 sq. in., find the area of the second.

19. In ex. 18 how do the arcs of the two sectors compare? Find the number of degrees in each arc.

Exercises — Group 93

THEOREMS

In any circle show that:

1. $d = c \div \pi$. **2.** $r = \sqrt{K \div \pi}$.

3. The area of a circle equals 4 times the area of a circle described on its radius as a diameter.

4. If in a circle two chords intersect at right angles and circles are constructed on the four segments of the chords as diameters, the area of the given circle will equal the sum of the four circles. (See ex. 9, p. 270.)

* **5.** Prove that the area included between two concentric circles of radii R and r is equal to $\pi R^2 - \pi r^2$.

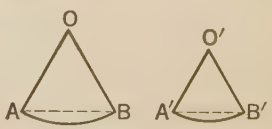
This area is called a *circular ring*.

6. The area of a circular ring equals the area of a circle whose diameter is a chord of the outer circle tangent to the inner circle.

* **7.** Similar segments are to each other as the squares of their radii.

SUG. $\dfrac{\text{Sector } OAB}{\text{Sector } O'A'B'} = \dfrac{r^2}{r'^2} = \dfrac{\triangle OAB}{\triangle O'A'B'}$.

Then use §§ 333, 337, etc.



8. In the regular pentagon $ABCDE$, if diagonal BE cuts AC and AD in P and Q respectively, prove: $\triangle APB \sim \triangle ABC$.

9. The perpendicular bisector of a side of a regular pentagon passes through the opposite vertex and bisects the angle at that vertex.

Exercises — Group 94

NUMERICAL

1. The sides of two regular octagons are 4 in. and 6 in. respectively. Find the ratio of their perimeters and of their areas.

2. Find the ratio of the areas of two regular hexagons whose sides are $2\frac{3}{4}$ in. and $4\frac{1}{8}$ in.

3. The sides of two regular polygons of the same number of sides are 1.8 in. and 4.5 in. The area of the first is 96.72 sq. in. Find the area of the second.

4. The perimeters of two regular hexagons are 30 in. and 72 in. What is the ratio of their apothems? of their areas?

5. If the length of an arc of 75° is 18 in., what is the radius?

6. The length of a circumference is 142 in. Find the length of an arc of 80°.

7. In a circle whose radius is 15 in. an arc is 48 in. long. How many degrees are there in the arc?

8. What is the effect on the circumference of a circle when the radius is multiplied by 3? by m? divided by 3? by m?

9. What is the effect on the area of a circle when the radius is multiplied by 3? by m? divided by 3? by m?

10. Find the circumference of a circle whose area is K.

If the radius of a circle is 9 *in., find the area of a circle:*

11. 4 times as large. **12.** 6 times as large. **13.** m times as large.

Find the area of a circle if the radius is:

14. $2r$ in. **15.** $\frac{a}{2}$ ft. **16.** $a\sqrt{2}$ ft. **17.** $\frac{2}{3}a\sqrt{3}$ in.

18. If the radius of one circle is 6 times the radius of another, what is the ratio of their areas? of their circumferences? of their diameters?

19. What effect does it have on the area of a circle if we double the radius?

20. What effect does doubling the area have on the radius of a circle?

21. Find the area included between two concentric circles whose radii are 4 in. and $6\frac{1}{2}$ in. respectively.

22. The radius of a circle is R. Express in terms of R the radius of a circle (*a*) 2 times as large; (*b*) $\frac{1}{2}$ as large; (*c*) m times as large.

23. The area of a circle is K. Express in terms of K the radius of a circle (*a*) twice as large; (*b*) $\frac{1}{2}$ as large.

24. Find the radius of a circle whose area equals the sum of the areas of two circles with radii 3 in. and 4 in. respectively.

25. If the area of one circle is 9 times the area of another and the radius of the first is 6 ft., find the radius of the second circle.

26. There are two circles. The diameter of the first is 12 in. and the area of the second is $2\frac{1}{4}$ times the area of the first. What is the diameter of the second circle?

27. A circle and a square each have a perimeter of 140 ft. Which contains the greater area, and how much greater is it?

28. The circumference of a circle is 15 in. Find the circumference of a circle having 3 times the area of the given circle.

29. Find the radius of a semicircle equal to an equilateral triangle whose side is 6 in.

If $r = 16$ in., find the length of an arc of the inscribed:

30. Regular hexagon.

31. Regular triangle.

32. Regular quadrilateral.

33. Regular octagon.

34. Two tangents form with each other an angle of 60°. Find the lengths of the arcs into which their points of contact divide the circle if the radius is 20 in.

35. A circle has an area of 72 sq. in. Find the length of an arc of 80°.

36. The length of a circle is 148 ft. Find the area of a sector whose angle is 40°.

37. Find the radius of the circle in which a sector of 60° contains 625 sq. in.

38. The area of a given sector equals the square constructed on the radius. Find the angle of the sector.

39. In a circle find the area of a segment whose chord is $12\sqrt{3}$ in. long and 6 in. from the center.

40. Find the area of a segment of a circle if the radius is 12 in. and the center of the chord is 18 in. from the center of the circle.

41. In a circle whose radius is 8 in. two parallel chords are drawn, each equal to the radius. Find the area bounded by the circle and the two chords.

42. What is the width of the ring between two concentric circles whose lengths are 120 ft. and 80 ft.?

* **43.** Find the number of degrees in the arc of a circle whose length equals the radius of the circle.

The corresponding angle at the center is called a *radian.*

44. A circular park 600 ft. in diameter has a drive 24 ft. wide around it. Find the area of the drive.

45. The diagonals of a rhombus are 16 in. and 24 in. Find the area of the inscribed circle.

46. The sides of a triangle are 9 in., 10 in., and 17 in. Find the area of the inscribed circle. (See ex. 9, p. 267.)

47. The sides of a triangle are 12 in., 13 in., and 5 in. Find the radius of the circle whose area equals the area of the triangle.

48. The radius of a circle is 12 in. What are the radii of the two circles, concentric to the given circle, which divide the given circle into three equal parts?

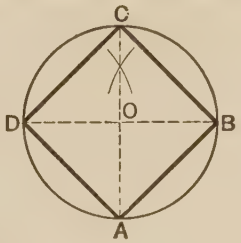
49. From a point without a circle whose radius is 20 in. two tangents are drawn. Find the area bounded by the two tangents and the minor arc which they intercept if the tangents include an angle of 120°.

50. Divide a circle into three parts in the ratio 1 : 2 : 3.

51. Find the diameter of the circle whose area and circumference have the same numerical measure.

Proposition IX. Problem

460. *To inscribe a square in a given circle.*



Given: Circle O.

Required: To inscribe a square in circle O.

Analysis:

1. See § 427. How many divisions will there be at O?
2. How will they be made and why will they be equal?
3. Why is $ABCD$ equilateral? regular?

Construction and Proof:

Left to the pupil.

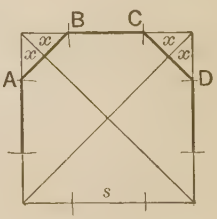
461. Corollary. *To inscribe in a given circle a regular polygon of* 8, 16, 32 $\cdots$ 2^n *sides.*

Exercises — Group 95

* **1.** To circumscribe a square about a given circle.

* **2.** To circumscribe about a given circle a regular polygon of 8, 16, 32 $\cdots$ 2^n sides.

* **3.** To inscribe a regular octagon in a given square.



Sug. $AB = BC = CD \cdots$.

$\therefore x^2 + x^2 = (AB)^2 = (BC)^2 = (s - 2x)^2$.

$\therefore s - 2x = x\sqrt{2}$, or $x = s - \dfrac{s\sqrt{2}}{2}$.

That is, x = one side of sq. $- \frac{1}{2}$ a diag.

Why is each angle 135°?

Find the area of a square inscribed in a circle if:

4. $r = 12$ in. **6.** $d = 4.6$ yd. **8.** $d = 5\,a$ ft.

5. $r = 6$ ft. **7.** $r = 6\sqrt{2}$ in. **9.** $r = a\sqrt{3}$ yd.

Find one side of a square inscribed in a circle if:

10. $K = 64$ sq. in. **13.** $K = 72.56$ sq. in.

11. $K = 16$ sq. ft. **14.** $K = 324\,\pi$ sq. ft.

12. $K = 6.25$ sq. yd. **15.** $K = a\sqrt{2}\,\pi$ sq. yd.

16. In ex. 4 to 9 find the perimeter and the apothem of each square.

17. In ex. 10 to 15 find the apothem and the radius of each square.

* **18.** The side of a square inscribed in a circle is equal to the radius multiplied by $\sqrt{2}$.

* **19.** The apothem of an inscribed square is equal to one half the radius multiplied by $\sqrt{2}$.

* **20.** The area of an inscribed square is $2\,R^2$.

* **21.** The area of a circumscribed square is $4\,r^2$.

22. The area of a circle is 576 sq. ft. Find the area of the inscribed square.

23. The radius of a circle is 8 in. Find the perimeter and the area of the regular inscribed octagon.

24. A square 12 in. on a side has its corners cut off so as to form a regular octagon. Find the area and the perimeter of the octagon.

* **25.** In a regular inscribed octagon $a = R\sqrt{2 - \sqrt{2}}$, where a is one side of the octagon and R is the radius of the circle.

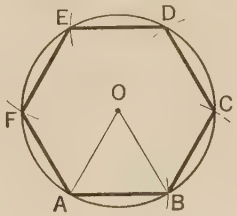
* **26.** In a regular circumscribed octagon one side is $2\,r(\sqrt{2} - 1)$, where r is the radius of the circle.

27. Each side of an equilateral triangle is 12 in. Find the area included between the sides and the inscribed circle.

28. If on the sides of an inscribed square as diameters semicircles are described, the sum of the areas of the four crescents lying outside the circle equals the area of the inscribed square.

Proposition X. Problem

462. *To inscribe a regular hexagon in a given circle.*



Given: Circle O.

Required: To inscribe a regular hexagon in circle O.

Analysis:

1. How many divisions will there be at O? What is the size of each? Why?
2. How can such an angle be constructed at O?
3. This locates two vertices. How are the others located?
4. Why is $ABCDEF$ equilateral? Why regular?

Construction and Proof:

Left to the pupil.

463. Corollary 1. *To inscribe in a given circle a regular polygon of* 12, 24, 48 $\cdots$ 3×2^n *sides.*

464. Corollary 2. *To inscribe a regular (equilateral) triangle in a given circle.*

465. Corollary 3. *The side of an inscribed regular hexagon is equal to the radius of the circle.*

Discussion:

1. Must all six angles at O be constructed? Explain.
2. If, starting at B, points are marked off successively on the circle, using AB as a radius, why will the last point fall on A?

Exercises — Group 96

* **1.** To circumscribe a regular hexagon about a given circle.

* **2.** To circumscribe about a given circle a regular polygon of 12, 24, 48 $\cdots 3 \times 2^n$ sides.

* **3.** To circumscribe a regular triangle about a given circle.

Find the apothem and perimeter of an inscribed regular hexagon if:

4. $r = 12$ in. **5.** $d = 2\sqrt{3}$ ft. **6.** $r = k$ yd.

Find the area of a regular hexagon inscribed in a circle if:

7. $r = 20$ in. **8.** $d = 4.6$ yd. **9.** $r = a\sqrt{2}$ ft.

Find one side of a regular hexagon whose area is:

10. $36\sqrt{3}$ sq. in. **11.** 72 sq. ft. **12.** 96.4 sq. yd.

Find the apothem, the perimeter, and the area of a regular hexagon inscribed in a circle whose area is:

13. $144\,\pi$ sq. in. **14.** 128 sq. yd. **15.** 48.6 sq. ft.

16. Solve ex. 4 to 6 for the circumscribed regular hexagon.

17. Solve ex. 7 to 9 for the circumscribed regular hexagon.

18. Solve ex. 13 to 15 for the circumscribed regular hexagon.

Find the apothem and perimeter of an inscribed equilateral triangle if:

19. $r = 24$ in. **20.** $d = 12.6$ yd. **21.** $r = 4\,n$ ft.

22. Solve ex. 19 to 21 for the circumscribed equilateral triangle.

Find the area of an equilateral triangle inscribed in a circle if:

23. $r = 48$ in. **24.** $r = 7.6$ yd. **25.** $d = 8\sqrt{3}$ ft.

Find the apothem and one side of an equilateral triangle whose area is:

26. $81\sqrt{3}$ sq. in. **27.** 128 sq. ft. **28.** 48.8 sq. yd.

Find the apothem, the perimeter, and the area of an equilateral triangle inscribed in a circle whose area is:

29. $1.44\,\pi$ sq. in. **30.** $75\,\pi$ sq. ft. **31.** $3\,m^2$ sq. yd.

32. Solve ex. 23 to 25 for the circumscribed equilateral triangle.

33. Solve ex. 29 to 31 for the circumscribed equilateral triangle.

34. Show how to cut off the corners of an equilateral triangle so as to form a regular hexagon.

* **35.** A side of an inscribed equilateral triangle is $R\sqrt{3}$.

* **36.** The apothem of an equilateral triangle is one half its radius.

* **37.** The apothem of a regular hexagon is $\frac{1}{2} R\sqrt{3}$.

38. One side of a circumscribed regular hexagon is $\frac{2}{3} r\sqrt{3}$.

39. One side of a circumscribed equilateral triangle is $2r\sqrt{3}$.

* **40.** The area of a regular inscribed hexagon is $\frac{3}{2} R^2\sqrt{3}$.

41. The area of a regular circumscribed hexagon is $2r^2\sqrt{3}$.

* **42.** The area of an inscribed equilateral triangle is $\frac{3}{4} R^2\sqrt{3}$.

43. The area of a circumscribed equilateral triangle is $3r^2\sqrt{3}$.

Exercises — Group 97

Miscellaneous Theorems

1. The area of a regular inscribed hexagon is the mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

* **2.** One side of a regular duodecagon is $R\sqrt{2-\sqrt{3}}$.

* **3.** The area of a regular inscribed duodecagon equals 3 times the square of the radius.

4. The diagonals joining the alternate vertices of a regular hexagon form another regular hexagon whose area is one third the area of the given hexagon.

5. The square inscribed in a semicircle is two fifths the square inscribed in the whole circle.

6. The area of a regular inscribed octagon is equal to the product of a side of an inscribed square and the diameter.

7. The areas of two triangles inscribed in equal circles are to each other as the products of their three sides.

Sug. Use ex. 6, p. 248, and § 404.

8. Prove that a square inscribed in a given circle is greater than any other parallelogram inscribed in that circle.

Exercises — Group 98

MISCELLANEOUS CONSTRUCTIONS

* **1.** Construct a circumference equal in length to the sum of two given circumferences.

2. Construct a circumference equal in length to the difference between two given circumferences.

* **3.** Construct a circle whose area equals the sum of the areas of two given circles.

4. Construct a circle whose area equals the difference between the areas of two given circles.

5. Construct a circle whose area equals 3 times the area of a given circle.

6. Construct a semicircle equal to a given circle.

SUG. $\pi r^2 = \frac{1}{2}\pi x^2$. $\therefore x^2 = 2r^2 = r^2 + r^2$.

In a given circle construct a segment containing:

7. An angle of 30°. **8.** An angle of 45°.

9. Construct three circumferences in the ratio 2 : 3 : 5.

* **10.** Construct a regular hexagon, given one of its sides.

* **11.** Construct a regular octagon, given one of its sides.

SUG. What is the size of one of its exterior angles?

12. In a given circle, construct a polygon similar to a given regular polygon.

13. Construct a regular hexagon equal to one third of a given regular hexagon. (See ex. 4, p. 302.)

14. Show how to inscribe a square in a sector whose angle is a right angle.

15. Construct a circle bisecting the area of a given circle and concentric with it.

* **16.** Construct a segment of a circle similar to two given similar segments and equal to their sum.

SUG. See ex. 7, p. 294.

17. Inscribe a square in a given segment of a given circle.

18. Inscribe a circle in a given sector.

19. In a given equilateral triangle construct three equal circles each tangent to the other two circles and to a side of the given triangle.

20. In a given circle construct three equal circles each tangent to the other two circles and to the given circle.

Exercises — Group 99

NUMERICAL

1. In a circle whose area is K find the apothem and the area of the inscribed square.

2. A square and a regular octagon are inscribed in a circle whose radius is 12 in. Find the difference between: (*a*) their perimeters; (*b*) their areas.

3. What is the area of a regular hexagon if its radius is r?

4. In a circle whose radius is 8 in. find the difference between the perimeters of the inscribed regular hexagon and the inscribed equilateral triangle.

5. An equilateral triangle, a square, and a regular hexagon, each has an area of 128 sq. in. What is the perimeter of each?

6. Find the areas of an equilateral triangle, a square, a regular hexagon, and a circle the perimeter of each of which is 36 in.

7. The area of a circle is 324 sq. in. If a square is circumscribed about this circle, find the area included between the circle and sides of the square.

8. The altitude of an inscribed equilateral triangle is to a side of the square circumscribed about the circle as 3 is to 4.

9. In a given circle whose radius is 12 in. a regular hexagon is inscribed. Find:

(*a*) The area of one segment whose chord is a side of the hexagon.

(*b*) The area of the triangle whose vertices are three consecutive vertices of the hexagon.

(*c*) The area of the triangle formed by joining alternate vertices of the hexagon.

(*d*) The area of the ring bounded by the given circle and the circle inscribed in the hexagon.

10. Find the radius of the circle if the difference of the sides of the regular inscribed and circumscribed equilateral triangles is 18 ft.

11. Find the radius of the circle if the sum of the areas of the regular inscribed and circumscribed hexagons is 144 sq. in.

Exercises — Group 100

Applications

1. If the minute hand of a tower clock is 5′ long, over what distance will its end point move in 24 minutes?

2. Three pumps with diameters 6″ each are used to keep a mine free from water. If one pump were used to do the same work what would its diameter have to be if the other conditions were the same?

3. A pulley 20″ in diameter makes 350 revolutions a minute (R.P.M.). Find the speed of the pulley belt, deducting 1% for slipping.

4. A certain increase in temperature causes a circular sheet of steel 3′ in diameter to increase in diameter by .4%. By how much is the area of the sheet increased?

5. Find the pressure on a safety valve $2\frac{3}{4}$ inches in diameter, assuming that the pressure of steam in the boiler is 125 pounds a square inch.

6. If the drive wheels of a locomotive are 60″ in diameter, find the number of revolutions a minute necessary to go 30 miles an hour.

7. Find the weight of the iron hoops on a tank 18′ in diameter, there being 16 hoops weighing 3 pounds per linear foot.

8. An approximate rule for determining the length of an arc of a circle is $s = 8w - \frac{W}{3}$, where s = length of arc, W = length of the chord of the arc, and w = length of the chord of half the arc. Find the error when $r = 8''$ and the angle $= 30°$.

9. A shaft to which a 5-inch pulley is attached makes 800 R.P.M. What size of pulley must be used to reduce this speed to 300 revolutions?

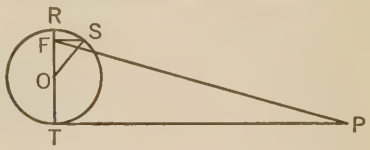
10. Find the length of the curved portion of a railroad track that connects two straight portions of track forming an angle of 60°, knowing that the curved portion is an arc of a circle of radius 150′ tangent at the ends of the straight tracks.

11. If two half dollars were melted and made into a single circular coin of the same thickness, what would be the ratio of the diameter of the new coin to that of the old one?

12. At the end of one side of a barn 36′ on a side a horse is tied by a rope 50′ long. Find the area over which he can graze.

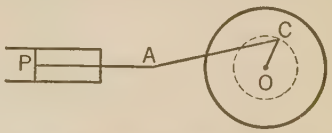
13. With a circle whose radius is 12″ determine the error in using the following rule for finding the approximate length of a circle:

Draw $TP \perp$ diameter RT and equal to 3 times RT. Draw OS so that $\angle SOR = 30°$. Draw $SF \perp TR$. Then FP equals the length of the circle.



14. In laying out a running track it was necessary to determine the radius of the quadrants to be formed at each corner of the rectangular field 300′ × 200′ so that the track would take 4 laps for a mile. Find the radius of each quadrant if they are all equal.

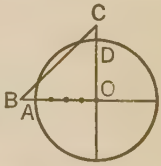
15. The distance btweeen the center of the crank pin C and the center of the flywheel O is 18″. What is the length of the stroke of piston P?



16. Show how to find by means of a carpenter's square the diameter of a circle having the same area as the sum of two given circles.

17. The following rule is used by sheet metal workers to obtain a square whose area equals the area of a given circle:

Divide the radius OA into 4 equal parts, place one of these parts from A to B and another from D to C, A and D being ends of two perpendicular diameters. Connect B and C. Then BC is one side of the square. Determine the error when $r = 8''$.



18. What is the waste in cutting the largest possible circular plate from a piece of sheet steel 16″ × 20″?

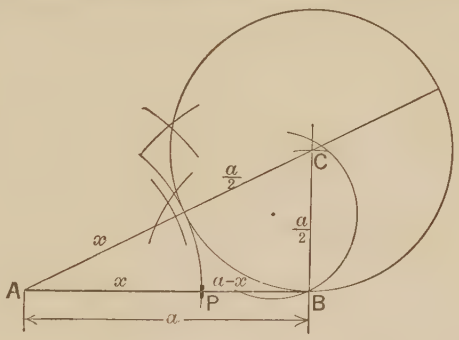
SUPPLEMENTARY PROPOSITIONS AND EXERCISES

466. A point divides a line in *mean and extreme ratio* (also called the *golden section*) when one segment is the mean proportional between the whole line and the other segment.

Thus, in the figure, P divides the line AB in mean and extreme ratio if $AB : AP = AP : PB$.

Proposition I. Problem

467. *To divide a given line segment in mean and extreme ratio.*



Given: Line a.

Required: To construct a line x such that $a : x = x : a - x$.

Analysis:

1. Show that $\left(x + \frac{a}{2}\right)^2 = a^2 + \left(\frac{a}{2}\right)^2$. (See illustration 1, p. 244.)

2. Then construct a line $= x + \frac{a}{2}$ (see § 369) and hence construct line x.

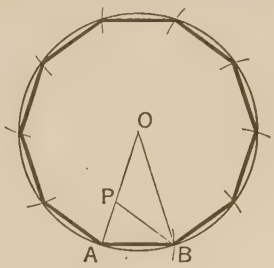
Construction and Proof:

Left to the pupil.

468. Corollary. *The value of x is approximately* .618 a.

Proposition II. Problem

469. *To inscribe a regular decagon in a given circle.*



Given: Circle O.

Required: To inscribe a regular decagon in circle O.

Analysis:

1. If AB is one side of the regular decagon, what must be the size of $\angle AOB$? of $\angle A$? of $\angle ABO$? Why?
2. The problem then is to construct on OA a $\triangle$ having these $\measuredangle$.
3. Such a $\triangle$ can be divided into two isosceles $\triangle$ by the bisector of $\angle B$. Explain. What lines then are equal?
4. Show that if point P is located, the $\triangle$ could be constructed.
5. What $\triangle$ are similar? What proportions follow?
6. Show that one proportion reduces to $AO : PO = PO : AP$.
7. Hence, how is point P located? (See § 467.)

Construction and Proof:

Left to the pupil.

470. Corollary 1. *To inscribe a regular pentagon in a given circle.*

How many equal divisions of the circle should there be? How can they be obtained from the diagram?

471. Corollary 2. *To inscribe in a circle a regular polygon of* 10, 20, 40 $\cdots$ 5×2^n *sides.*

472. Corollary 3. *To inscribe in a circle a regular pentadecagon, that is, a regular polygon of* 15 *sides.*

What must be the size of each $\angle$ at the center?

How can this $\angle$ be obtained from the central angles found in §§ 462 and 469?

Construct this $\angle$ at O and hence determine one side.

Ex. A regular polygon of 15×2^n sides may be constructed.

Exercises — Group 101

1. To circumscribe a regular decagon about a given circle.

2. To circumscribe about a given circle a regular polygon of 10, 20, 40 $\cdots$ 5×2^n sides.

3. To circumscribe a regular pentagon about a given circle.

* **4.** In a regular inscribed decagon one side is $\frac{R}{2}(\sqrt{5} - 1)$.

* **5.** In a regular inscribed pentagon one side is $\frac{R}{2}\sqrt{10 - 2\sqrt{5}}$.

6. The lines joining in order the mid-points of the radii of a regular pentagon inclose a regular pentagon whose area equals one fourth the area of the given pentagon.

* **7.** Prove that any two diagonals of a regular pentagon, not drawn to the same vertex, divide each other in mean and extreme ratio.

* **8.** If all the diagonals of a regular pentagon are drawn, their intersections will be the vertices of another regular pentagon.

* **9.** Construct the following angles: 36°; 18°; 9°; 72°; 24°.

* **10.** To construct a regular pentagon having given one side.

Sug. Use ex. 9 or § 387.

11. In a circle whose radius is $1\frac{3}{4}$ in., construct a segment that shall contain an angle of 36°.

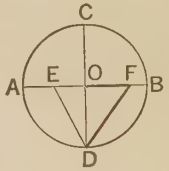
12. To divide a right angle into five equal parts.

* **13.** Construct a regular pentagon, given a diagonal.

Sug. See ex. 7 above; also ex. 10, p. 284.

14. Prove that the following method for determining the side of a regular inscribed decagon and the side of a regular inscribed pentagon in any given circle is correct:

In circle O draw AB and CD, two perpendicular diameters. Bisect AO and join E with D. Take $EF = ED$ and join F with D.



Prove: OF is a side of the regular inscribed decagon and DF is a side of the regular inscribed pentagon.

Sug. Show $OF = \frac{R}{2}(\sqrt{5} - 1)$. (See ex. 4, p. 309.)

Show $DF = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}$. (See ex. 5, p. 309.)

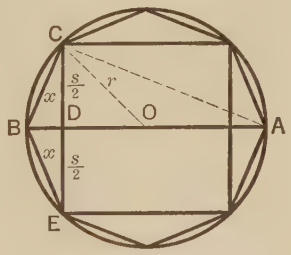
473. Historical Note. In §§ 460, 462, 469 and the corollaries it has been shown that a circle may be divided into 2×2^n, 3×2^n, 5×2^n, and 15×2^n equal parts, n being any positive integer or zero (excepting in the case of 2×2^n, where n could not be zero), and that regular polygons with corresponding number of sides may be constructed. For centuries it was believed that these were the only regular polygons that could be constructed with ruler and compasses only. In the latter part of the 18th century Gauss (1777–1855), a German mathematician, devised a method for constructing a regular polygon of 17 sides using the ruler and compasses only. Since then several other constructions for a regular polygon of 17 sides have been discovered.

Hence, if n = the number of sides of the polygon, then for $n = 20$ or $n < 20$, regular polygons can be constructed with ruler and compasses if $n = 3, 4, 5, 6, 8, 10, 12, 15, 17, 20$; but it has been proved that regular polygons cannot be so constructed if $n = 7, 9, 11, 13, 14, 18, 19$.

474. Finding an Approximate Value of π. This requires finding an approximate value of the circumference of a circle, which is done by finding the perimeters of successive inscribed regular polygons each having twice as many sides as the preceding one. The work will be facilitated by Prop. III.

Proposition III. Problem

475. *Given the radius of a circle and the side of a regular inscribed polygon, to derive a formula for the side of a regular inscribed polygon of double the number of sides.*



Given: Circle O, with radius r, CE a side of a regular inscribed polygon, denoted by s, and CB a side of the regular inscribed polygon of double the number of sides, denoted by x.

Required: To express x in terms of s and r.

Analysis:

1. Show that $x^2 = 2r \cdot BD = 2r(r - OD)$ (§ 367, III)

$$= 2r\left(r - \sqrt{r^2 - \frac{s^2}{4}}\right).$$

2. Hence show $x = \sqrt{\dfrac{2r(2r - \sqrt{(2r)^2 - s^2})}{2}}$

$$= \sqrt{r(2r - \sqrt{4r^2 - s^2})}$$

Proof: Left to the pupil.

476. Corollary. *If the diameter of the circle is* 1, *then*

$$x = \sqrt{\frac{1 - \sqrt{1 - s^2}}{2}} = \frac{1}{2}\sqrt{2 - 2\sqrt{1 - s^2}}$$

477. Inscribe a square in a circle whose diameter is 1, and compute the length of one side and of the perimeter of the square. Then, using § 476, find in succession the length of one side and of the perimeter of regular inscribed polygons of 8, 16, 32 $\cdots$ 2^n sides. The following table shows the results up to and including 2^{12}:

Number of Sides	Length of One Side	Length of Perimeter
4	.707107 $\cdots$	2.828427 $\cdots$
8	.382683 $\cdots$	3.061467 $\cdots$
16	.195090 $\cdots$	3.121445 $\cdots$
32	.098017 $\cdots$	3.136548 $\cdots$
64	.049068 $\cdots$	3.140331 $\cdots$
128	.024541 $\cdots$	3.141277 $\cdots$
256	.012272 $\cdots$	3.141513 $\cdots$
512	.005136 $\cdots$	3.141572 $\cdots$
1024	.003068 $\cdots$	3.141587 $\cdots$
2048	.001534 $\cdots$	3.141591 $\cdots$
4096	.000767 $\cdots$	3.141592 $\cdots$
$\cdots$	$\cdots$	$\cdots$

478. Since the diameter of the circle used in computing the above table was 1, the numerical measure of its circumference is π (see § 446), and hence it is apparent that the last column gives successive approximations to the value of π. This approximation, to the nearest fourth decimal place, gives $\pi = 3.1416$. In practice, however, where extreme accuracy is not required, the approximation $\pi = 3\frac{1}{7}$ is sufficiently accurate.

479. Historical Note. The attempt to find the exact value of π is directly connected with one of the most famous problems of antiquity, namely, the *quadrature of the circle*, or *squaring the circle*. This problem requires the finding of a side of the square whose area equals the area of the circle. Since the area of a circle is equal to one half the product of its circumference and radius,

a square of equal area could be constructed if the exact circumference could be found from the radius. To accomplish this in turn it would be necessary to have the exact ratio between the circumference and the diameter of the circle, that is, the exact value of π.

Hippocrates (430 B.C.), a Greek geometrician, attempted to square the circle by means of exercise 28, page 299. That is, having shown that a square could be constructed equal to a crescent bounded by a quadrant and the semicircle constructed on the chord of the quadrant, he constructed a half regular hexagon in a semicircle and drew semicircles on the sides. Then, assuming from this exercise that a square could be constructed equal to a crescent bounded by *any* two semicircles, he proceeded to show that a rectilinear figure could be obtained equal to the semicircle, from which a square could be constructed equal to the circle.

This problem of squaring the circle gave rise early to various attempts at finding the value of π. The earliest reference to π is by Ahmes, an Egyptian, about 1550 B.C. He used a formula for the area of a circle, which gave a value to π, when expressed in our notation, of 3.1604. The people of Babylon and the early Hebrews used 3 for the value of π. (See I Kings vii. 23 and II Chronicles iv. 2.)

Archimedes (287–212 B.C.), one of the greatest mathematicians, by a method similar to that used in § 477 found that π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$, that is, between 3.1428 and 3.1408. Other later mathematicians computed the value of π correct to 9, 17, and 35 decimal places.

With the invention of the Calculus by Newton (1642–1727), methods for computing π far simpler than the method of inscribed polygons were discovered. By the use of these more modern methods the value of π has been computed to 707 decimal places.

While the study of the computation of π is important and interesting as a bit of theory, yet the endless list of figures to which its value may be found is wholly unnecessary, for Professor Newcomb has shown that "ten decimal places are sufficient to give the circumference of the earth to a fraction of an inch."

Below is given the value of π computed to 30 decimal places:

$$\pi = 3.141,592,653,589,793,238,462,643,383,379 \cdots$$

Exercises — Group 102

NUMERICAL — LINES AND ANGLES

1. In the isosceles triangle ABC ($CA = CB$), angle A is trisected by lines AD and AE. If angle ADC is 64°, find all the angles.

2. A set of exterior angles of a triangle are in the ratio 2 : 5 : 8. What are the angles of the triangle?

3. The difference between two adjacent angles of a parallelogram is 72°. Find the angles of the parallelogram.

4. Two parallel lines are cut by a transversal. If one of the angles formed is 64°, find all the other angles.

5. An exterior angle of a regular polygon is 24°. How many sides has the polygon?

6. Three consecutive sides of an inscribed quadrilateral intercept arcs of 40°, 100°, and 75°. Find the angle at each vertex, the angle between the diagonals, and the angle between opposite sides produced to meet.

7. In a circle whose radius is 15 in. find the angle at the center subtended by an arc 10 in. long.

8. Find the central angle of a sector whose perimeter equals the circumference.

9. In the semicircle $ABCDE$ chord BD is parallel to the diameter AE and bisects radius OC at right angles.

Prove: $\overset{\frown}{AB} = 30°$ and $\overset{\frown}{BC} = 60°$.

10. One side of a rhombus is 8 in. and one angle is 60°. Find the two diagonals.

11. Find the length of an arc of 75° in a circle whose radius is 10 in.

12. If the length of an arc of a quadrant is 8 ft., what is the radius?

13. If $r = 16$ in., find the length of an arc of a regular inscribed pentagon.

14. Find the apothem and the perimeter of a regular inscribed hexagon if the diameter is 6.4 yd.

15. Find the apothem and the perimeter of a regular inscribed triangle if the radius is $6\sqrt{3}$ ft.

16. The base of a triangle is 18 in. Two lines are drawn parallel to the base, terminating in the other two sides and dividing the triangle into three equal parts. Find the lengths of these lines.

17. In a triangle whose base is 18 in. and altitude 15 in., two lines are drawn parallel to the base and trisecting the altitude. Find the lengths of these parallel lines.

18. If the chord of a quadrant of a circle is 16 in., find the radius.

19. How many revolutions are made by a wheel whose radius is $2\frac{1}{2}$ ft. in rolling 45 yd.?

20. What are the lengths of the longest and the shortest chord drawn through a point 12 in. from the center of a circle whose radius is 13 in.?

21. How far from the center is a chord 72 in. long in a circle whose radius is 45 in.?

22. A pole broke off 16 ft. from the ground, the top striking the ground 30 ft. from the base. How high was the pole?

23. The legs of a right triangle are in the ratio 3 : 4. If the hypotenuse is 45 in., how long are the legs?

24. The three sides of a triangle are in the ratio 3 : 4 : 5 and the difference between the shorter sides is 8 in. Find the sides. What kind of triangle is it? Why?

25. The hypotenuse of a right triangle is 24 cm. and the altitude on the hypotenuse is 12 cm. Find (*a*) the segments of the hypotenuse made by the altitude and (*b*) the legs.

26. Two sides of a triangle are 34 in. and 20 in. and the altitude on the third side is 16 in. Find the third side.

27. Find one side of an equilateral triangle if the altitude is 16 in.

28. Find the altitude of an equilateral triangle if one side is 24 cm.

29. Find one side of a square inscribed in a circle whose radius is 12 in.

30. Two circles have radii 13 in. and 3 in. and their centers are 26 in. apart. Find the length (*a*) of a common external tangent; (*b*) of a common internal tangent.

31. The sides of a triangle are 16 in., 24 in., and 30 in. Find the segments of the longest side made by the bisector of (*a*) the opposite interior angle; (*b*) the opposite exterior angle.

32. The sides of a triangle are 15 units, 20 units, and 25 units. Find the length of (*a*) the three medians; (*b*) the altitude on the longest side. What kind of triangle is it? Why?

33. Find the length of a circle circumscribed about a rectangle whose sides are 40 cm. and 42 cm.

34. Find the radius of a circle whose circumference equals the perimeter of an equilateral triangle whose altitude is 12 in.

35. Two chords intersect within a circle. One chord 21 in. long is divided in the ratio 1 : 6; the other chord is divided in the ratio 2 : 3. Find the segments of each chord.

36. Through a point 11 in. from the center of a circle whose radius is 13 in. a chord 14 in. long is passed. What are the segments of the chord?

37. From a given point a tangent and a secant are drawn to a circle. Find the length of the tangent if the internal and the external segments of the secant are 8 in. and 24 in.

38. Find the length of the shortest line that can be drawn from the end of a tangent 8 in. long to a circle whose radius is 6 in.

39. From a point 9 in. from a circle whose radius is 12 in. a secant 27 in. long is drawn. What are the segments of the secant?

40. An equilateral triangle is inscribed in a circle whose radius is 12 in. Find the lengths of the two chords parallel to the base which trisect the altitude.

41. The rock of Gibraltar is 1440 ft. high. A lighthouse 60 ft. high situated on this rock is just visible to a ship at sea. Assuming the radius of the earth to be 4000 mi., how far at sea is the ship?

SUG. Assume that the chord and its arc are equal.

42. The sides of a triangle are 17 in., 8 in., and 15 in. Find (*a*) the bisector of the largest angle; (*b*) the diameter of the circumscribed circle.

43. The bases of a trapezoid are 12 in. and 8 in. and the altitude is 6 in. Find the altitudes of the two triangles formed by producing the legs to meet.

44. In ex. 43, find the altitudes of the two vertically opposite triangles formed by the diagonals.

45. The sides of a triangle are 73 cm., 52 cm., and 35 cm. Find: (*a*) the altitude on the side of 73 cm.; (*b*) the bisector of the angle opposite the side of 35 cm.; (*c*) the median to the side of 52 cm.

46. If $\sqrt{x+2} : \sqrt{x} = 1 : 5$, find x.

47. By the use of addition transform and solve:
(*a*) $3a - 7 : 7 = 4 : 3$; (*b*) $5 : 3x - 5 = 2x - 8 : 8$.

48. By the use of subtraction transform:

$$3a + \sqrt{2a} - 5 : \sqrt{2a} - 5 = 7a + \sqrt{3a} + 2 : \sqrt{3a} + 2.$$

Exercises — Group 103

NUMERICAL — AREAS

1. Find the area of the triangle in ex. 45 above.

2. Find the area of an isosceles triangle whose base is 24 in. and each of whose legs is 13 in.

3. Find the area of an equilateral triangle whose side is 12 cm.

4. Find the area of an equilateral triangle whose altitude is 6 in.

5. Find the area of a right isosceles triangle if the altitude on the hypotenuse is 4 cm.

6. Find the area of a right triangle if one leg is 21 in. and the hypotenuse is 29 in.

7. Find the area of a right triangle if one leg is 10 in. and the altitude on the hypotenuse is 6 in.

8. Find the area of a square whose diagonal is 8 cm.

9. If the area of an equilateral triangle is $25\sqrt{3}$ sq. in., find one side.

10. If the area of an equilateral triangle is 48 sq. cm., find the altitude.

11. Find the area of a triangle whose base is 24 in. and altitude 18 in.

12. In ex. 11, how far from the base must a parallel to the base be drawn to cut off a triangle equal to one half the given triangle?

13. In ex. 11, what is the area of the triangle cut off by a line parallel to the base and bisecting the altitude?

14. Find the area of the rhombus in ex. 10, p. 314.

15. Find the area of a rhombus one of whose sides is 58 in. and one of whose diagonals is 80 in.

16. Find the area of an isosceles trapezoid whose bases are 8 in. and 20 in. and whose legs are 10 in. each.

17. In ex. 16, find the area included between the lower base and a line parallel to the base and 3 in. above the base.

18. Two sides of a triangle are 18 in. and 10 in. and the included angle is 60°. Find the area.

19. One triangle has sides 6 in. and 8 in., a second triangle has sides 5 in. and 15 in. If the angles included between these sides are equal and the area of the first triangle is 96 sq. in., what is the area of the second triangle?

20. Two similar triangles have a combined area of 78 sq. cm. If two corresponding sides are 6 cm. and 9 cm., find the two areas and the two corresponding altitudes.

21. The dimensions of a rectangle are in the ratio 3 : 5 and the area is 135 sq. yd. Find the dimensions.

22. Find one side of a regular hexagon whose area is $48\sqrt{3}$ sq. ft.

23. Find the radius of a circle whose area is $154\,\pi$ sq. in.

24. Find the area of a circle whose circumference is 220 m.

25. Find the area of a circular sector if its arc is 18° and the radius is 4 ft.

26. Find the radius of a circle in which a sector of 75° contains 64 sq. ft.

27. What is the radius of a circle whose area is doubled when the radius is increased by 3 in.?

28. Two circles have areas A^2 and K^2. If the radius of the first is R, what is the radius of the second?

29. Find the radius of a circle equal to a square whose side is 8 in.

30. Find the area of a regular inscribed triangle if $d = 16$ ft.

31. Find the apothem, the perimeter, and the area of a regular triangle inscribed in a circle whose area is 324π sq. in.

32. Find the apothem, the perimeter, and the area of a regular hexagon inscribed in a circle whose area is 96 sq. m.

33. The side AB of an inscribed regular hexagon and the side DC of an inscribed square are parallel and on the same side of the center. Find the area included between them if $r = 8$ in.

34. In ex. 8, p. 309, show that the areas of the two pentagons are in the ratio $2 : (7 - 3\sqrt{5})$.

35. AB, BC, and CA are each arcs of 60°, the radius in each case being 12 in. Find the area bounded by the three arcs.

36. Find the area of a circular sector, the chord of half the arc being $12\sqrt{2 - \sqrt{3}}$ in. and the radius 12 in.

37. Find the radius of a circle equal to the combined areas of three circles whose radii are 8 in., 12 in., and 18 in.

38. In a circle whose radius is 24 in. find the area of a segment whose angle is (*a*) 60°; (*b*) 150°.

39. In a circle whose radius is 20 cm. find the angle of a sector whose area is 60 sq. cm.

40. A concrete walk around a fountain is in the form of a circular ring whose inner and outer diameters are 60 ft. and 90 ft. How many square feet are there in the walk?

41. Two tangents to a circle from the same point include an angle of 60°. If the radius is 8 in., find the area bounded by the two tangents and the minor arc which they intercept.

42. The diameter of a circle is 12 in. What are the radii of the circles concentric to the given circle which divide it into four equal parts?

43. A cow is tied to the corner post of a pasture which is in the form of a rectangle. How long must the rope be if the cow is to graze over one fourth of an acre?

Exercises — Group 104

Miscellaneous Theorems

1. If the two equal sides of an isosceles triangle are extended the same distance through the vertex, the line joining the two end points is parallel to the base.

2. When any two lines are cut by a transversal, the difference between any pair of corresponding angles formed is equal to the angle between the two lines.

3. If the altitudes from two vertices of a triangle are drawn, the angle between them is the supplement of the angle at the third vertex of the triangle.

4. If two circles are tangent internally or externally at A, and any line through A cuts the circle in B and C, tangents at B and C are parallel.

5. In ex. 4, prove that diameters through B and C are parallel.

6. Two tangents are drawn to circle O from point P. PO produced meets the circle at M and the tangent at M meets the other two tangents (extended) at R and S.

Prove: $PR = PS$.

7. If from any point P within an equilateral triangle perpendiculars are drawn to the sides, the sum of these perpendiculars equals the altitude of the triangle.

Sug. Through P draw a parallel to one side.

8. Each side of an equilateral triangle is trisected. Prove that if all these points are joined in order, a regular hexagon is formed.

9. Prove that twice the sum of the medians of a triangle is less than 3 times the perimeter.

10. A triangle is equal to a quadrilateral if the triangle has 2 sides and the included angle equal respectively to the 2 diagonals of the quadrilateral and the angle between them.

11. If from one end of a diameter a chord is drawn, a perpendicular to the chord from the other end of the diameter meets the chord on the circle.

12. If two circles intersect in P and if tangents at P to the two circles meet the circles in Q and R, prove that $PQ : PR$ is in the same ratio as the radii of the two circles.

13. If from the ends of the longer base of a trapezoid lines are drawn to the mid-point of the other base and extended to meet the legs produced, the line joining the points of intersection is parallel to the two bases. (See §§ 345 and 365.)

14. ABC is an isosceles triangle ($CA = CB$). If CA is produced to D so that $AD = CA$, prove: $(DB)^2 = (CA)^2 + 2(AB)^2$.

15. $ABCD$ is a parallelogram and P is any point in DC. If AP and BC produced meet in E, prove: $\triangle DPE = \triangle PBC$.

Sug. $\triangle DAE = \frac{1}{2}\square ABCD = \triangle ADP + \triangle BCP$, etc.

16. If a diameter of a circle is divided into two parts and upon these parts semicircles are constructed on opposite sides of the diameter, the semicircles will divide the given circle into two parts having the same ratio as the parts of the diameter.

17. The square of a side of the regular inscribed triangle equals 3 times the square of a side of the regular inscribed hexagon.

18. If a point is taken within a regular polygon, prove that the sum of the distances to the sides of the polygon is equal to the radius of the inscribed circle multiplied by the number of sides of the polygon.

19. The radius of an inscribed regular polygon is the mean proportional between its apothem and the radius of a similar circumscribed polygon.

20. $ABCD$ is an inscribed quadrilateral. AD and BC produced meet in E, and AB and DC produced meet in F. If the bisectors of angles E and F meet in O, prove that angle FOE is a right angle.

21. In ex. 20, prove that the lines joining in order the points of intersection of the circle and the bisectors of angles E and F form a rhombus.

22. $ABCD$ is a parallelogram. If perpendiculars are dropped to a line from the four vertices, prove that the sum of the perpendiculars from A and C equals the sum of the perpendiculars from B and D.

23. If either opposite pair of triangles formed by the diagonals of a quadrilateral are equal in area, the sides of the quadrilateral which are not included are parallel.

24. Squares AD and BE are constructed on sides AC and BC of triangle ABC.

Prove: $AE = BD$.

25. An equiangular polygon inscribed in a circle is regular if the number of sides is odd.

26. An equilateral polygon circumscribed about a circle is regular if the number of sides is odd.

Exercises — Group 105

Miscellaneous Constructions

1. Construct a right triangle, given the segments of the hypotenuse made by the altitude.

2. Construct a right triangle, given an acute angle and the sum of the hypotenuse and a leg.

3. Construct a triangle, given a, b, m_c.

4. Construct a triangle, given A, B, R.

5. Construct a triangle, given m_a, m_b, m_c.

Sug. Extend m_c one third its length.

6. Construct a rhombus, given one angle and a diagonal.

7. Construct a circle touching three given lines, two being parallel.

8. Construct a circle touching a given line at a given point and passing through another given point. (See §§ 311 and 305.)

9. A line segment of given length (l) moves always parallel to a given fixed line (k) and has one end in a given circle. Construct the locus of the other end of the line.

10. Construct the locus of the centers of all circles of a given radius which touch a given circle.

11. Construct the locus of all points equidistant from two given equal circles. (See § 312.)

12. From a given point outside a given circle draw a secant that shall be bisected by the circle. Discuss.

13. On the line of centers of two circles O and O' determine the point P from which tangents to the two circles are equal.

Sug. Let $OO' = d$, $OP = x$, $PO' = d - x$, and the radii be R and r. Then $x^2 - R^2 = (d - x)^2 - r^2$, whence $d(2x - d) = (R + r)(R - r)$. Determine $2x - d$ and then x.

14. In ex. 13, to what does x reduce when (*a*) the circles are equal? (*b*) the two circles are tangent internally? (*c*) the two circles are tangent externally?

15. In ex. 13, show that the line through P perpendicular to OO' is the locus of all points from which tangents to the two circles are equal.

16. Construct the locus of the mid-points of parallel chords of a circle.

17. Construct the locus of the mid-points of chords drawn from the same point in a given circle.

18. Construct the locus of the vertex of a given angle whose sides pass through two given points.

19. Construct a regular decagon, given one side.

20. Construct two lines, having given their sum (m) and their product (n^2), where m and n are two line segments.

Sug. See § 385. Here AB = the sum and $(CD)^2$ = the product.

21. Construct a circle equal to three fifths of a given circle.

22. $ABCD$ is a parallelogram and R is any point in CD. Construct the locus of P, the mid-point of lines parallel to BR and terminated by the sides of the parallelogram.

23. Given two points C and D on the same side of line AB. Find a point P in line AB, such that $\angle APC = \angle BPD$.

Sug. See ex. 13, p. 68.

FORMULAS USED IN PLANE GEOMETRY

In the formulas given below the following notation will be used:

$a, b, c, \cdots$, sides of polygon $ABC\cdots$	p, perimeter
$s = \frac{1}{2}(a + b + c)$	K, area
h_c, altitude on side c	r, radius of a circle
m_c, median on side c	c, circumference of a circle
t_c, bisector of $\angle C$	$\pi = 3\frac{1}{7}$ or 3.1416, approximately
h, altitude of $\triangle$, etc.	d, diameter of a circle, diagonal of a square
b, base of $\triangle$, etc.	a, apothem of a regular polygon
p^b_c, projection of b on c	

b and b' = bases of a trapezoid

Length of Lines

1. *Right triangle* ($\angle C$ being the rt. $\angle$): $c^2 = a^2 + b^2$ (§ 369)
2. *Equilateral triangle:* $h = \frac{a}{2}\sqrt{3}$ (Ex. 2, p. 267)
3. *Square:* $d = a\sqrt{2}$ (Ex. 1, p. 267); $a = \sqrt{K}$ (Ex. 5, p. 267)
4. $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$ (Ex. 10, p. 267)
5. $m_c = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$ (Ex. 5, p. 247)
6. $t_c = \frac{2}{a+b}\sqrt{abs(s-c)}$ (Ex. 9, p. 248)
7. $a^2 = b^2 + c^2 \mp 2cp^b_c$ according as $\angle A$ is acute or obtuse (Ex. 1, 2, pp. 246, 247)
8. $p^b_c = \frac{b^2 + c^2 - a^2}{2c}$ when $\angle A$ is acute. (Ex. 3, p. 247)
9. In *similar polygons:* $a : a' = b : b' \cdots$ (§ 350);
   $p : p' = a : a' \cdots$ (§ 379)
10. In *regular polygons:* $p : p' = r : r' = a : a'$ (§ 435)
11. In *circles:* $c : c' = r : r' = d : d'$ (§§ 442, 443)
12. In *circles:* $c = \pi d$, or $c = 2\pi r$ (§ 446)
13. An *arc of a circle* $= \frac{\text{central angle}}{180^\circ} \times \pi r$ (§ 447)
14. $S_{2n} = \sqrt{r(2r - \sqrt{4r^2 - S^2_n}}$ (§ 475)

Areas of Plane Figures

1. *Rectangle:* $K = bh$ (§ 393)
2. *Square:* $K = a^2$ (Ex. 5, p. 267)
3. *Parallelogram:* $K = bh$ (§ 399)
4. *Triangle:* $K = \frac{1}{2}bh$ (§ 403)
5. *Triangle:* $K = \sqrt{s(s-a)(s-b)(s-c)}$ (Ex. 9, p. 267)
6. *Equilateral triangle:* $K = \frac{a^2}{4}\sqrt{3}$ (Ex. 4, p. 267)
7. *Trapezoid:* $K = \frac{h}{2}(b + b')$ (§ 408)
8. *Regular polygon:* $K = \frac{1}{2}ap$ (§ 438)
9. *Circle:* $K = \pi r^2$, or $K = \frac{1}{4}\pi d^2$ (§ 452)
10. *Sector of a circle:* $K = \frac{1}{2}r \times$ arc (§ 457)

$$\text{or } K = \frac{\text{central angle}}{360^\circ} \times \pi r^2 \text{ (§ 456)}$$

11. *Segment of a circle:* $K =$ sector $- \triangle$ formed by the two radii and the chord of the segment (§ 458).
12. In any two *similar plane figures:*

$$K : K' = a^2 : a'^2 \text{ and } a : a' = \sqrt{K} : \sqrt{K'} \quad \text{(§ 412)}$$

13. In two *circles:*

$$K : K' = r^2 : r'^2 = d^2 : d'^2$$
$$\text{and } r : r' = d : d' = \sqrt{K} : \sqrt{K'} \quad \text{(§ 453)}$$

TABLES

Table of Linear Measures

12	inches (in. *or* ″)	= 1 foot (ft. *or* ′)
3	feet	= 1 yard (yd.)
5½	yards	= 1 rod (rd.)
320	rods	= 1 mile (mi.)

Table of Square Measures

144	square inches (sq. in.)	= 1 square foot (sq. ft.)
9	square feet	= 1 square yard (sq. yd.)
30¼	square yards	= 1 square rod (sq. rd.)
160	square rods	= 1 acre (A.)

Table of Metric Linear Measures

10 *millimeters* (mm.)	= 1 *centimeter* (cm.)	=	.4	in. (nearly)
10 centimeters	= 1 decimeter (dm.)	=	4.	in. (nearly)
10 decimeters	= 1 *meter* (m.)	=	39.37	in.
10 meters	= 1 dekameter (Dm.)	=	10.9	yd.
10 dekameters	= 1 hektometer (Hm.)	=	109.	yd.
10 hektometers	= 1 *kilometer* (Km.)	=	.6	mi.

Table of Metric Square Measures

100 *square millimeters*	= 1 *square centimeter*	=	.154	sq. in.
100 square centimeters	= 1 square decimeter	=	10.8	sq. ft.
100 square decimeters	= 1 *square meter*	=	1.19	sq. yd.
100 square meters	= 1 square dekameter *	=	119.6	sq. yd.
100 square dekameters	= 1 square hektometer †	=	2.5	A.
100 square hektometers	= 1 square kilometer	=	.4	sq. mi.

* Usually called an *are*.

† Usually called a *hektare*.

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